(Cont'd) Idea (of the reduction to unconstrainted problem)

$$F(\vec{x}, \lambda) = F(x_1, \cdots, x_n, \lambda) \quad \text{is of } n+1 \text{ variables}$$
$$= f(\vec{x}) - \lambda (g(\vec{x}) - c)$$

critical pts :

$$\underset{\mathbb{R}^{n+1}}{\underset{\cap}{\vec{0}}} = \underset{(n+1)\text{-variables}}{\underset{\uparrow}{\vec{\nabla} F}} = \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial \lambda} \right)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x_i} = \dfrac{\partial f}{\partial x_i} - \lambda \dfrac{\partial g}{\partial x_i} & \forall i = 1, \cdots, n \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(g - c) \end{cases}$$

n-variables

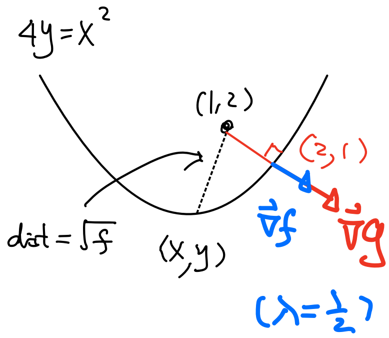
$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x_i} = \lambda \dfrac{\partial g}{\partial x_i} & \forall i = 1, \cdots, n \quad (\Leftrightarrow \vec{\nabla} f = \lambda \vec{\nabla} g) \\ g = c \end{cases}$$

$\Rightarrow$ extreme of f under the constraint $g = c$

by Lagrange multipliers Thm. #

eg 2 (cont'd) minimize $f(x,y) = (x-1)^2 + (y-2)^2$

under constraint $g(x,y) = x^2 - 4y = 0$



Soln: Consider

$$F(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - 0)$$
$$= (x-1)^2 + (y-2)^2 - \lambda(x^2 - 4y)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = 2(x-1) - 2\lambda x & \text{——— (1)} \\ 0 = \dfrac{\partial F}{\partial y} = 2(y-2) + 4\lambda & \text{——— (2)} \\ 0 = \dfrac{\partial F}{\partial \lambda} = -(x^2 - 4y) & \text{——— (3)} \end{cases}$$

(2) $\Rightarrow$ $2\lambda = 2 - y$

Put into (1) $\Rightarrow$ $0 = 2(x-1) - (2-y)x$

$\Rightarrow$ $y = \dfrac{2}{x}$

Put into (3) $\Rightarrow$ $x^2 - \dfrac{8}{x} = 0$ $\Rightarrow$ $x^3 = 8$ $\Rightarrow$ $x = 2$

hence $y = 1$

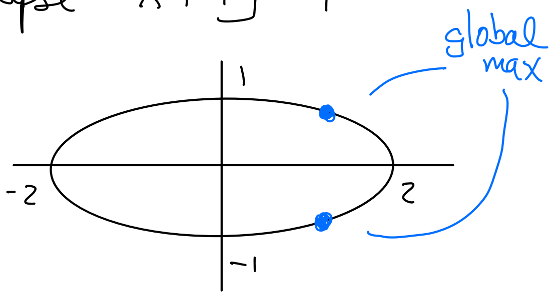
$\therefore$ $(2,1)$ is the only critical point.

$\Rightarrow$ f has a minimum at $(2,1) \in g^{-1}(0)$

with value $f(2,1) = (2-1)^2 + (1-2)^2 = 2$ ※

eg 3 Maximize $f(x,y) = xy^2$ on the ellipse $x^2+4y^2=4$

Soln:
$$\begin{cases} f(x,y) = xy^2 \\ g(x,y) = x^2+4y^2 \end{cases}$$



& hence consider

$$F(x,y,\lambda) = xy^2 - \lambda(x^2+4y^2-4)$$

$$\begin{cases} 0 = \dfrac{\partial F}{\partial x} = y^2 - 2\lambda x \\ 0 = \dfrac{\partial F}{\partial y} = 2xy - 8\lambda y \\ 0 = \dfrac{\partial F}{\partial \lambda} = 4 - x^2 - 4y^2 \end{cases}$$

(Ex!) By "simple" calculation, we have

$$(x,y) = (\pm 2, 0) \text{ or } \left(\pm\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right)$$

$$\left(= \begin{matrix} (2,0) \\ (-2,0) \end{matrix} \quad \& \quad \begin{matrix} (\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}) & (\frac{2}{\sqrt{3}}, -\sqrt{\frac{2}{3}}) \\ (-\frac{2}{\sqrt{3}}, \sqrt{\frac{2}{3}}) & (-\frac{2}{\sqrt{3}}, -\sqrt{\frac{2}{3}}) \end{matrix} \right)$$

are all critical points of the problem.

Comparing values of f at all these 6 critical pts:

$$f(\pm 2, 0) = 0$$

$$f\left(\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{\sqrt{3}} \cdot \frac{2}{3} = \frac{4}{3\sqrt{3}} \longleftarrow \text{max}$$

$$f\left(-\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{2}{\sqrt{3}} \cdot \frac{2}{3} = -\frac{4}{3\sqrt{3}} \longleftarrow \text{min}$$

$\therefore$ For $f(x,y)$ on $g(x,y)=4$, the

global max value $= \frac{4}{3\sqrt{3}}$ at $\left(\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

(global min value $= -\frac{4}{3\sqrt{3}}$ at $\left(-\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$)

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<u>eg 1 (cont'd)</u> Using Lagrange multiplier, find global max/min of

$$f(x,y) = x^2+2y^2-x+3 \quad \text{on} \quad x^2+y^2=1$$

(Step 2 of the original global max/min problem on $x^2+y^2\le 1$)

<u>Soln</u>: Let $\begin{cases} f(x,y) = x^2+2y^2-x+3 \\ g(x,y) = x^2+y^2 \end{cases}$

& $F(x,y,\lambda) = x^2+2y^2-x+3-\lambda(x^2+y^2-1)$

$$\begin{cases} 0 = \frac{\partial F}{\partial x} = 2x-1-2\lambda x & \text{——— (1)} \\ 0 = \frac{\partial F}{\partial y} = 4y-2\lambda y & \text{——— (2)} \\ 0 = \frac{\partial F}{\partial \lambda} = -(x^2+y^2-1) & \text{——— (3)} \end{cases}$$

(2) $\Rightarrow$ $(2-\lambda)y=0$

$\Rightarrow$ $y=0$ or $\lambda=2$

If $y=0$, then (3) $\Rightarrow$ $x^2=1$ $\Rightarrow$ $x=\pm 1$

Hence $(x,y)=(\pm 1, 0)$ are critical pts.

If $y\neq 0$, then $\lambda = 2$

$\therefore$ (1) $\Rightarrow$ $2x - 1 - 2(2)x = 0$

$\Rightarrow$ $x = -\frac{1}{2}$

Put into (3) $(\frac{1}{2})^2 + y^2 = 1 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

$\therefore$ $(x,y) = (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ are critical pts.

All together, the critical pts. are

$$(x,y) = (\pm 1, 0),\ (-\tfrac{1}{2}, \pm\tfrac{\sqrt{3}}{2})$$

Comparing values $f(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{21}{4}$ $\leftarrow$ max (on $x^2+y^2=1$)

$f(1, 0) = 3$ $\leftarrow$ min (on $x^2+y^2=1$)

$f(-1, 0) = 5$

$\therefore$ Global max. of f on $\{x^2+y^2=1\} = \frac{21}{4}$ at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$

Global min. of f on $\{x^2+y^2=1\} = 3$ at $(1, 0)$

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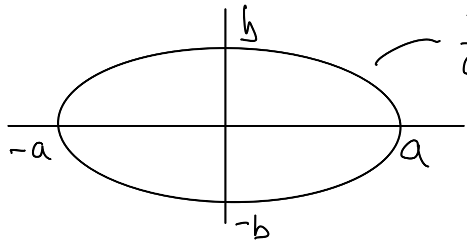
Classification of Quadratic Constraints

2-variables : $g(x,y) = Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F$
(Conic Section)

Typical examples for level curve $g(x,y) = c$

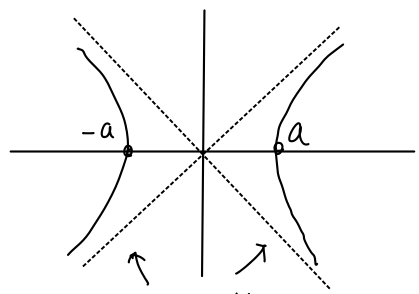
(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a, b > 0$ **Ellipse** (**circle** if $a=b$)



$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

closed & bounded

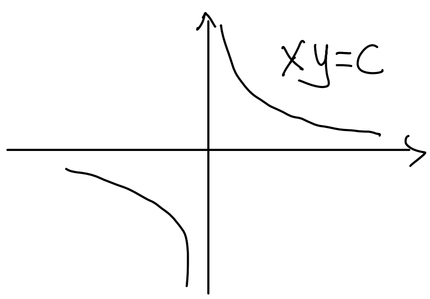
(ii) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $a, b > 0$ **Hyperbola**



$\frac{x}{a} = \pm \frac{y}{b}$

unbounded

($xy = c$, $c \neq 0$, is also a hyperbola)

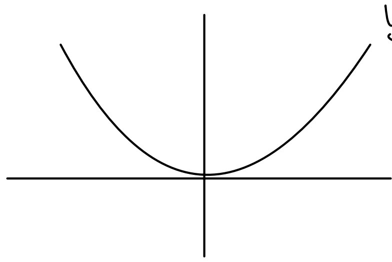


$xy = c$ $(c > 0)$

$$\left(x = \pm a \sqrt{1 + \frac{y^2}{b^2}} = \pm \frac{a\,y}{b} \sqrt{1 + \frac{b^2}{y^2}} \sim \pm \frac{a}{b} y \text{ as } |y| \to +\infty \right)$$

(iii') $y = ax^2, \quad a \neq 0$ <u>Parabola</u>

(only 1 quadratic term)



<u>unbounded</u>

(iv) <u>Degenerate Cases</u> $(a>0, b>0)$

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \longrightarrow$ a point $(0,0)$
- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 \longrightarrow$ empty set
- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \longrightarrow \dfrac{x}{a} = \pm \dfrac{y}{b}$ a pair of intersecting lines

  $(xy = 0)$
- $x^2 = c \longrightarrow x = \pm\sqrt{c}$
  - a pair of parallel lines if $c>0$
  - a "double" line if $c=0$
  - empty set if $c<0$

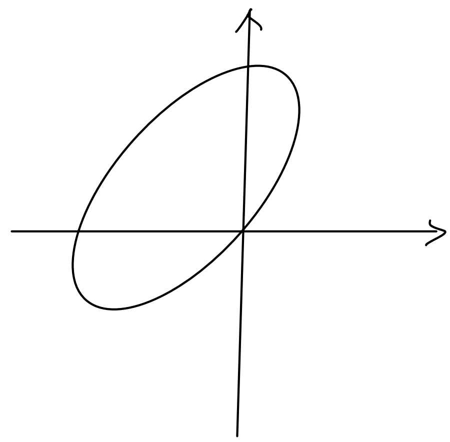
<u>Fact</u>: By a change of coordinates, any <u>quadratic constraint</u> $g(x,y) = c$ can be transformed to one of the form above.

(<u>Proof</u>: Omitted)

Hence level sets of quadratic constraints are ellipse, hyperbola, parabola, & degenerated cases.

eg $17x^2 - 12xy + 8y^2 + 16\sqrt{5}x - 8\sqrt{5}y = 0$

$$\Leftrightarrow \quad \frac{u^2}{1^2} + \frac{v^2}{2^2} = 1$$



where
$$\begin{cases} u = \dfrac{2x-y}{\sqrt{5}} + 1, \\ v = \dfrac{x+2y}{\sqrt{5}} \end{cases}$$

(Check!)

Remark: Ellipse is closed and bounded $\Rightarrow$ Any continuous $f(x,y)$ restricted to an ellipse has global max & min.

Not the case for hyperbola & parabola.