Lecture 11: Recall:

Goal: Develop iterative method: find a sequence $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ such that $\vec{x}_k \to \vec{x}^* =$ sol. of $A\vec{x} = \vec{f}$ as $k \to \infty$.

Remark: We can stop when error is small enough.

Method: Splitting method

Consider a linear system $A\vec{x} = \vec{f}$ where $A \in M_{n\times n}$ (n is BIG)

Split A as follows: $A = N + (A - N) = N - \underbrace{(N - A)}_{P}$

$= N - P$

Then: $A\vec{x} = \vec{f} \iff (N - P)\vec{x} = \vec{f} \iff N\vec{x} = P\vec{x} + \vec{f}$

Develop an iterative scheme as follows:

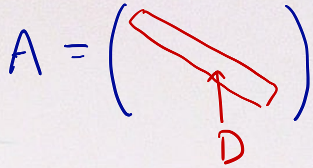
$$(*) \quad N\vec{x}^{n+1} = P\vec{x}^{n} + \vec{f}$$

If $\{\vec{x}^n\}_{n=1}^{\infty}$ converges, then it converges to the sol $\vec{x}^*$ of $A\vec{x} = \vec{f}$

<u>Remark:</u>

- N should be simple : easy to find inverse.
- N should have an inverse
- N should be "related to" A.
- N should be chosen such that $\{\vec{x}^n\}_{n=1}^{\infty}$ converges.

Splitting choice 1: Jacobi method



Take $N = D$ = diagonal part of A

Split A as $A = D - (D - A)$

Then: $A\vec{x} = \vec{f} \iff D\vec{x} - (D-A)\vec{x} = \vec{f}$

$$\iff D\vec{x} = (D-A)\vec{x} + \vec{f}$$

We can consider an iterative scheme such that:

$$D\vec{x}^{k+1} = (D-A)\vec{x}^{k} + \vec{f}$$

$$\iff \vec{x}^{k+1} = D^{-1}(D-A)\vec{x}^{k} + D^{-1}\vec{f}$$

Remark: All diagonal entries of A must be non-zero, such that D is non-singular.

This is equivalent to solving: (assume $A = (a_{ij})_{1\le i,j\le n}$) $\vec{x}^k = \begin{pmatrix} x_1^k \\ \vdots \\ x_n^k \end{pmatrix}$

$$\begin{cases} a_{11}x_1^{k+1} + a_{12}x_2^k + a_{13}x_3^k + \dots + a_{1n}x_n^k = f_1 & (\text{for } x_1^{k+1}) \\ a_{21}x_1^k + a_{22}x_2^{k+1} + a_{23}x_3^k + \dots + a_{2n}x_n^k = f_2 & (\text{for } x_2^{k+1}) \\ \vdots \\ a_{n1}x_1^k + a_{n2}x_2^k + a_{n3}x_3^k + \dots + a_{nn}x_n^{k+1} = f_n & (\text{for } x_n^{k+1}) \end{cases}$$

Example: Consider: $\begin{pmatrix} 5 & -2 & 3 \\ -3 & 9 & 1 \\ 2 & -1 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

Then: Jacobi method gives:

$$\vec{x}^{k+1} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 2 & -3 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}\vec{x}^k + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Start with $\vec{x}^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The sequence almost converge in 7 iteration to get: $\vec{x}_7 = \begin{pmatrix} -0.186 \\ 0.331 \\ -0.423 \end{pmatrix}$

Question:

- Does it always converge?
- Is the initialization important?

Splitting choice 2: Gauss - Seidel method

Split A as $A = L + D + U$

lower tri diagonal upper tri

Develop an iterative scheme : $L\vec{x}^{k+1} + D\vec{x}^{k+1} + U\vec{x}^{k} = \vec{f}$

(So, take $N = L + D$ and $P = -U$)

$\Leftrightarrow \quad \vec{x}^{k+1} = -(L+D)^{-1}U\vec{x}^{k} + (L+D)^{-1}\vec{f}$. (Solve lower-triangular linear system)

This is equivalent to:

$$\begin{cases} a_{11}x_1^{k+1} + a_{12}x_2^{k} + a_{13}x_3^{k} + \cdots + a_{1n}x_n^{k} = f_1 & (\text{for } x_1^{k+1}) \\ a_{21}x_1^{k+1} + a_{22}x_2^{k+1} + a_{23}x_3^{k} + \cdots + a_{2n}x_n^{k} = f_2 & (\text{for } x_2^{k+1}) \\ \vdots & \\ a_{n1}x_1^{k+1} + a_{n2}x_2^{k+1} + a_{n3}x_3^{k+1} + \cdots + a_{nn}x_n^{k+1} = f_n & (\text{for } x_n^{k+1}) \end{cases}$$

Remark: Again, all diagonal entries of A must be non-zero, in order that $L+D$ is non-singular.

Example 2: Continue with Example 1.

$$\vec{x}^{k+1} = -\begin{pmatrix} 5 & 0 & 0 \\ -3 & 9 & 0 \\ 2 & -1 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \vec{x}^{k} + \begin{pmatrix} 5 & 0 & 0 \\ -3 & 9 & 0 \\ 2 & -1 & 7 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Start with $\vec{x}^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. After 7 iterations, we get: $\vec{x}^7 = \begin{pmatrix} -0.186 \\ 0.331 \\ -0.423 \end{pmatrix}$

Question: Does Jacobi / Gauss-Seidel method always converge?

Example 3: Consider: $\begin{pmatrix} 1 & -5 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

Jacobi: $\vec{x}^{k+1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 5 \\ -7 & 0 \end{pmatrix} \vec{x}^{k} + \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^{-1} \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

$\vec{x}^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \ldots \quad \vec{x}^7 = \begin{pmatrix} -214374 \\ -300127 \end{pmatrix}$ (Doesn't converge)

Gauss-Seidel also doesn't converge!!

Analysis of convergence

Let $A = N - P$

Goal: Solve $A\vec{x} = \vec{f} \iff (N-P)\vec{x} = \vec{f}$

Consider the iterative scheme: $N\vec{x}^{m+1} = P\vec{x}^{m} + \vec{f}$, $m = 0, 1, 2, \ldots$

Let $\vec{x}^* =$ sol of $A\vec{x} = \vec{f}$. Define error: $\vec{e}_m = \vec{x}^m - \vec{x}^*$, $m = 0, 1, 2 \ldots$

Now, (1) $N\vec{x}^{m+1} = P\vec{x}^{m} + \vec{f}$

(2) $N\vec{x}^{*} = P\vec{x}^{*} + \vec{f}$ $\quad$ ($\because$ $A\vec{x}^* = \vec{f} \iff (N-P)\vec{x}^* = \vec{f}$)

(1)−(2): $N(\vec{x}^{m+1} - \vec{x}^*) = P(\vec{x}^{m} - \vec{x}^*)$

$$\iff N\vec{e}^{m+1} = P\vec{e}^{m} \iff \vec{e}^{m+1} = \overbrace{N^{-1}P}^{M}\vec{e}^{m}.$$

Let $M = N^{-1}P$. We get: $\vec{e}^{m} = M^{m}\vec{e}^{0}$

$\bar{A}$ssume a simple case : let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be the set of linearly independent eigenvectors of M ($\vec{u}_i$ can be complex-valued vectors)

(In other words, assume diagonalizable)

Let $\vec{e}^{\,0} = \sum_{i=1}^{n} a_i \vec{u}_i$. Then:

$$\vec{e}^{\,m} = M^m \vec{e}^{\,0} = \sum_{i=1}^{n} a_i M^m \vec{u}_i = \sum_{i=1}^{n} a_i \lambda_i^m \vec{u}_i$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are corresponding eigenvalues.

WLOG, we can assume:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda$$

Assume $|\lambda_1| < 1$. Then:

$$\vec{e}^{\,m} = \lambda_1^m \left\{ a_1 \vec{u}_1 + \sum_{i=2}^{n} a_i \left(\frac{\lambda_i}{\lambda_1}\right)^m \vec{u}_i \right\} \to 0 \text{ as } m \to \infty$$

Remark: • In order to reduce error by a factor of 10^{-m}, then we need about k iterations such that $|\lambda_1|^k < 10^{-m}$

That is, $k > \dfrac{m}{-\log_{10}(\rho(M))} := \dfrac{m}{R}$

Here, we call $\rho(M) = |\lambda_1|$ the asymptotic convergence factor or the spectral radius.

$\therefore$ $\rho(M) \stackrel{\text{def}}{:=} \max\limits_{k} \{ |\lambda_k| : \lambda_k = \text{eigenvalue of } M \}$ is a good indicator for convergence.

• Finding $\rho(M)$ is difficult $\Longrightarrow$ Numerically (next topic)