MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 8

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-11-08 (Friday) 23:59***.

1. Let $V = \mathbb{C}^3, S = \{(1, i, 0), (1 - i, 2, 4i)\} \subset V$.
   (a) Find an orthonormal basis for $\text{span}(S)$.
   (b) Extend S to get an orthonormal basis S' of V.
   (c) Let $x = (3 + i, 4i, -4)$. Prove that $x \in \text{span}(S)$.

2. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$.

3. Let V be the vector space of all sequence σ in F (where $F = \mathbb{R}$ or $\mathbb{C}$) such that $\sigma(n) \neq 0$ for only finitely many positive integers n. For $\sigma, \mu \in V$, we define

$$\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}.$$

   Since all but a finite number of terms of the series are zero, the series converges.
   (a) Prove that $\langle \cdot, \cdot \rangle$ in an inner product on V, and hence V is an inner product space.
   (b) For each positive integer n, let e_n be the sequence defined by $e_n(k) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta. Prove that $\{e_1, e_2, ...\}$ is an orthonormal basis for V.
   (c) Let $\sigma_n = e_1 + e_n$ and $W = \text{span}(\{\sigma_n : n \geq 2\})$.
      i. Prove that $e_1 \notin W$, so $W \neq V$.
      ii. Prove that $W^\perp = \{0\}$, and conclude that $W \neq (W^\perp)^\perp$.

4. Let V and $\{e_1, e_2, ...\}$ be defined as in Q3. Define $T : V \to V$ by

$$T(\sigma)(k) = \sum_{i=k}^{\infty} \sigma(i) \quad \text{for every positive integer } k.$$

   Note that the infinite series in the definition of T converges because $\sigma(i) \neq 0$ for only finitely many i.
   (a) Prove that T is a linear operator on V.
   (b) Prove that for any positive integer n, $T(e_n) = \sum_{i=1}^{n} e_i$.
   (c) Prove that T has no adjoint.

5. Prove that if $V = W \oplus W^\perp$ and T is the projection on W along $W^\perp$, then $T = T^*$.

Exercises

The following are extra recommended exercises not included in the homework.

1. In each of the following parts, find the orthogonal projection of the given vector on the given subspace W of the inner product space V. Then find the distance from the given vector to the subspace W.

   (a) $V = \mathbb{R}^2, u = (2, 6)$, and $W = \{(x, y) : y = 4x\}$.

   (b) $V = \mathbb{R}^3, u = (2, 1, 3)$, and $W = \{(x, y, z) : x + 3y - 2z = 0\}$.

   (c) $V = P(\mathbb{R})$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt, h(x) = 4 + 3x - 2x^2$, and $W = P_1(\mathbb{R})$.

2. Let β be a basis for a finite-dimensional inner product space.

   (a) Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$.

   (b) Prove that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$, then $x = y$.

3. Let $V = C([-1, 1])$. Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. Prove that $W_e^{\perp} = W_o$, where the inner product on V is defined by
$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt.$$

4. Let V be an inner product space over F. Prove the *polar identities*: For all $x, y \in V$,

   (a) $\langle x, y\rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$ if $F = \mathbb{R}$.

   (b) $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k\|x + i^k y\|^2$ if $F = \mathbb{C}$, where $i = \sqrt{-1}$.

5. Let A be an $n \times n$ matrix with complex entries. Prove that $AA^* = I$ if and only if the rows of A form an orthonormal basis of $\mathbb{C}^n$.

6. Let $V = F^n$ and let $A \in M_{n\times n}(F)$.

   (a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle$ for all $x, y, \in F$.

   (b) Suppose that for some $B \in M_{n\times n}(F)$, we have $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in V$. Prove that $B = A^*$.

   (c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

   (d) Define linear operators T and U on V by $T(x) = Ax$ and $U(x) = A^*x$. Show that $[U]_\beta = [T]_\beta^*$ for any orthonormal basis β for V.