Lecture 23:

Claim 3: $V = K_{\lambda_1} + K_{\lambda_2} + \dots + K_{\lambda_k}$

Pf: By M.I. on k = # of distinct eigenvalues.

When $k=1$, let m = multiplicity of λ_1. Then, char poly of T " $(\lambda_1 - t)^m$.

By Caley-Hamilton Thm, $g(T) = (\lambda_1 I - T)^m = 0 \leftarrow$ zero transf.

$$\therefore \quad K_{\lambda_1} = N((T - \lambda_1 I)^m) = V$$

$\therefore$ Thm is true for $k=1$.

Assume that the thm is true for any lin. op. w/ fewer than k distinct eigenvalues.

Consider $T: V \to V$ with k distinct eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_k$

Claim 4: Let $W = R\left((T-\lambda_k I)^m\right)$ ← multiplicity of λ_k

Then: (1) $T|_W : W \to W$ is well-defined. (exercise)

(2) $T|_W$ has $k-1$ distinct eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$

(3) $(T-\lambda_k I)^m|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$ is onto $(i<k)$

Assuming Claim 4 is true.

Let $\vec{x} \in V$. Then: $(T-\lambda_k I)^m \vec{x} \in W$

By induction hypothesis, $\exists\, \vec{w}_i \in K'_{\lambda_i}$ = generalized eigenspace of λ_i of $T|_W$.

such that $(T-\lambda_k I)^m \vec{x} = \vec{w}_1 + \vec{w}_2 + \ldots + \vec{w}_{k-1}$

Easy to check: $K'_{\lambda_i} \subseteq K_{\lambda_i}$ for $i<k$

$\because \quad (T-\lambda_k I)^m\big|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$ is onto, then:

for each $\vec{w}_i \in K_{\lambda_i}'$, $\exists\, \vec{v}_i \in K_{\lambda_i} \ni (T-\lambda_k I)^m(\vec{v}_i) = \vec{w}_i$

$$\therefore (T-\lambda_k I)^m(\vec{x}) = (T-\lambda_k I)^m(\vec{v}_1) + \cdots + (T-\lambda_k I)^m(\vec{v}_{k-1})$$

$$\Leftrightarrow (T-\lambda_k I)^m(\vec{x} - \vec{v}_1 - \vec{v}_2 - \cdots - \vec{v}_{k-1}) = \vec{0}.$$

$$\therefore \overbrace{\vec{x} - \vec{v}_1 - \vec{v}_2 - \cdots - \vec{v}_{k-1}}^{\vec{v}_k} \in N\left((T-\lambda_k I)^m\right) = K_{\lambda_k}$$

$$\therefore \vec{x} = \underset{K_{\lambda_1}}{\underset{\cap}{\vec{v}_1}} + \cdots + \underset{K_{\lambda_{k-1}}}{\underset{\cap}{\vec{v}_{k-1}}} + \underset{K_{\lambda_k}}{\underset{\cap}{\vec{v}_k}}$$

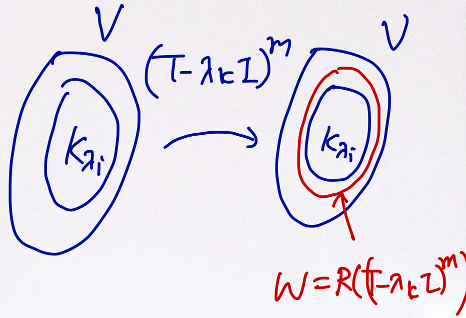
$$\therefore V = K_{\lambda_1} + \cdots + K_{\lambda_k}$$

By M.I, the thm is true.

Proof of Claim 4

Note: $(T-\lambda_k I)\big|_{K_{\lambda_i}}$ is 1-1 and onto (if $i<k$)

$\Rightarrow (T-\lambda_k I)^m\big|_{K_{\lambda_i}}$ is also onto



Also, $E_{\lambda_i} \subseteq K_{\lambda_i} \subseteq W = R((T-\lambda_k I)^m)$

$\therefore$ λ_i is an eigenvalue of $T|_W$ for $i<k$.

$\therefore$ $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ are eigenvalues of $T|_W$.

Next, suppose λ_k is an eigenvalue of $T|_W$.

Suppose $T|_W(\vec{v}) = \lambda_k \vec{v}$ for $\vec{v} \neq \vec{0}$ and $\vec{v} \in W = R((T-\lambda_k I)^m)$

Write $\vec{v} = (T-\lambda_k I)^m(\vec{y})$

$$\therefore \quad 0 = (T-\lambda_k I)\vec{v} = (T-\lambda_k I)(T-\lambda_k I)^m(\vec{y})$$
$$= (T-\lambda_k I)^{m+1}(\vec{y})$$

$$\Rightarrow \quad \vec{y} \in K_{\lambda_k} = N((T-\lambda_k I)^m)$$

$$\therefore \quad \underbrace{(T-\lambda_k I)^m(\vec{y})}_{\vec{v}} = \vec{0}$$

Contradiction.

Claim 5: Let β_i = ordered basis of K_{λ_i}.

Then: $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is a disjoint union and a basis of V.

Pf: **Disjoint union:** Let $\vec{x} \in \beta_i \cap \beta_j$ $(i \neq j) \subseteq K_{\lambda_i} \cap K_{\lambda_j}$.

$$(T - \lambda_i I)(\underset{\in K_{\lambda_j}}{\overset{\neq \vec{0}}{\vec{x}}}) \neq \vec{0} \qquad (\, T - \lambda_i I|_{K_{\lambda_j}} \text{ is } 1\text{-}1 \,)$$

$$(T - \lambda_i I)^2 (\vec{x}) \neq \vec{0}$$

$$\vdots$$

$$(T - \lambda_i I)^p (\vec{x}) \neq \vec{0} \quad \text{for } \forall p.$$

$\therefore$ $\vec{x} \notin K_{\lambda_i}$ (Contradiction)

$\therefore$ $\beta_i \cap \beta_j = \phi$.

Basis: Let $\vec{x} \in V$. By claim 3,

$$\vec{x} = \vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_k \quad \text{where} \quad \vec{v}_i \in K_{\lambda_i}$$

$\therefore$ $\vec{x}$ is a lin. comb. of vectors in $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$

$\therefore$ $V = \text{Span}(\beta)$

Let $q = |\beta|$. Then: $\dim(V) \leq q$

Let $d_i = \dim(K_{\lambda_i})$. Then: $q = \sum_i d_i \leq \sum_i m_i = \dim(V)$

$\therefore$ $q = \dim(V) \implies \beta$ is a basis.

<u>Claim 6:</u> $\dim(K_{\lambda_i}) = m_i$

<u>Pf:</u>

$$\sum_i d_i = \sum_i m_i \implies \sum_i \overset{\geq 0}{(m_i - d_i)} = 0$$

$$\sum_i d_i = \dim(V) = \sum_i m_i$$

$$\implies m_i = d_i \text{ for } \forall i.$$

$$V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \dots \oplus K_{\lambda_k}$$

<u>Claim 7</u>: Let $\gamma_1 = \{ (T - \lambda I)^{m_1} \vec{v}_1, \ldots, \vec{v}_1 \}$

$$\vdots$$

$$\gamma_q = \{ (T - \lambda I)^{m_q} \vec{v}_q, \ldots, \vec{v}_q \}$$

If initial vectors are linearly independent, then:

$\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_q$ is disjoint union and it's linearly independent.

<u>Pf</u>: <u>Disjoint union</u>: Easy, obvious. (exercise)

<u>Linear independence</u>: Use M.I. on n = # of element in γ.

When $n = 1$, trivial

Assume the thm is true for γ having less than n elements

When $|\gamma| = n$, let

$$\gamma_1' = \{(T-\lambda I)^{m_1}\vec{v}_1, \ \ldots, \ (T-\lambda I)\vec{v}_1\}$$

$$\vdots$$

$$\gamma_q' = \{(T-\lambda I)^{m_1}\vec{v}_q, \ \ldots, \ (T-\lambda I)\vec{v}_q\}$$

Let $\gamma' = \gamma_1' \cup \gamma_2' \cup \ldots \cup \gamma_q'$. $\therefore |\gamma'| = n-q$

Let $W = \mathrm{span}(\gamma)$. Let $U = (T-\lambda I)|_W : W \to W$

Then: $R(U) = \mathrm{span}(\gamma')$ (Check)

$\because$ initial vectors of γ_i's are L. I.

$\therefore$ γ' is L. I. (by induction hypothesis)

$\therefore$ $\dim(R(U)) = |\gamma'| = n-q$

Also, $S = \{ (T-\lambda I)^{m_1} \vec{v}_1, \ldots, (T-\lambda I)^{m_q} (\vec{v}_q) \} \subseteq N(U)$

$$U = (T-\lambda I)|_W$$

$\therefore \quad \dim(N(U)) \geq q$

$$\therefore \quad \underset{\underset{|\gamma|}{\shortparallel}}{n} \geq \dim(W) = \dim(R(U)) + \dim(N(U))$$

$$\geq n-q + q = n$$

$\therefore \quad \dim(W) = n, \quad |\gamma| = n$ and $\operatorname{span}(\gamma) = W$

$\therefore \quad \gamma$ is a basis of W and γ is L. I.

<u>Claim 8:</u> Suppose β = basis of V = disjoint union of cycles.

Then: ① For each cycle γ in β, $W = \operatorname{span}(\gamma)$ is T-invariant and $[T|_W]_\gamma$ = Jordan block.

② β = JC basis for V.

Claim 9: Let λ = eigenvalue of T.

Then: K_λ has a basis β = union of disjoint cycles w.r.t. λ.

Pf: By M.I. on $n = \dim(K_\lambda)$.

When $n=1$, trivial.

Suppose the result is true for $\dim(K_\lambda) < n$.

When $\dim(K_\lambda) = n$. Let $U = (T - \lambda I)\big|_{K_\lambda} : K_\lambda \to K_\lambda$.

Then: $\dim(R(U)) < \dim(K_\lambda) = n$

$$\left(\because \dim(K_\lambda) = \dim(\underset{E_\lambda}{\underset{\cup\!\!\!\mid}{N(U)}}) + \dim(R(U)) \quad \dim(E_\lambda) \geq 1 \right)$$

Let K_λ' = generalized eigenspace corresponding to λ of $T|_{R(U)} : R(U) \to R(U)$

Easy to check that $R(U) = K_\lambda'$ (Check) $K_\lambda' \subseteq R(U)$

By induction hypothesis, $\exists$ disjoint cycles $\gamma_1, \gamma_2, \ldots, \gamma_q$ of $T|_{R(U)} \Rightarrow$

$\gamma = \bigcup_{i=1}^{q} \gamma_i$ is a basis for $R(U) = K_\lambda'$

Let $\gamma_i = \{ (T|_{R(U)} - \lambda I)^{m_i} \vec{x}_i, \ldots, \vec{x}_i \}$

$\vec{x}_i \in K_\lambda' = R(U)$

Let $\vec{x}_i = U\vec{v}_i = (T - \lambda I)\vec{v}_i$, $\vec{v}_i \in K_\lambda$

Define: $\tilde{\gamma}_i = \{ \underbrace{\boxed{(T - \lambda I)^{m_i+1}(\vec{v}_i)}}_{\vec{w}_i}, \ldots, (T - \lambda I)(\vec{v}_i), \vec{v}_i \}$

Note: $\bigcup_{i=1}^{q} \gamma_i$ is L.I. $\therefore$ $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\}$ is L.I. subset of E_λ.

Extend S to a basis of E_λ: $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_s\}$

By construction, $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_q, \{\vec{u}_1\}, \{\vec{u}_2\}, \ldots, \{\vec{u}_s\}$ are disjoint unions of cycles $\ni$ initial vectors are L.I.

$\therefore \quad \tilde{\gamma} = \bigcup_{i=1}^{q} \tilde{\gamma}_i \cup \{\vec{u}_1\} \cup \{\vec{u}_2\} \cup \dots \cup \{\vec{u}_s\}$ is L.I. subset of K_λ.

Now, we show that $\tilde{\gamma}$ is a basis of K_λ.

Suppose $|\gamma| = r = \dim(R(U))$

γ basis of $R(U)$

Then: $|\tilde{\gamma}| = r + q + s$

$\therefore \quad \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q, \vec{u}_1, \dots, \vec{u}_s\}$ is a basis of E_λ

$$E_\lambda = N(U) = N((T - \lambda I)|_{K_\lambda})$$

$\therefore \quad \dim(N(U)) = q + s.$

$\therefore \quad \dim(K_\lambda) = \dim(R(U)) + \dim(N(U)) = r + (q + s) = |\tilde{\gamma}|$

$\therefore \quad \tilde{\gamma}$ is a basis for K_λ.

Claim 10: T has JCF.