Lecture 22:

Theorem: Any $A \in M_{n \times n}(\mathbb{C})$ is similar to a matrix of the following form:

$$J = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & & 0 \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} \lambda_2 & 1 & & 0 \\ & \lambda_2 & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} \lambda_N & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_N \end{matrix}} \end{pmatrix}$$

(Jordan Canonical Form of A)

$\lambda_1, \lambda_2, \ldots, \lambda_N$ are eigenvalues of A
(not necessarily distinct)

Remark: Jordan canonical consists of blocks in this form:

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \lambda \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in M_{k\times k}(\mathbb{C})$$

It is called Jordan block of size k with eigenvalue λ.

Prop: (1) A has only 1 eigenvalue λ (multiplicity is k)

(2) $\dim(E_\lambda) = 1$ ($\Rightarrow$ A is not diagonalizable if $k \neq 1$)

(3). The smallest positive integer p s.t.

$(A - \lambda I)^p = 0$ is equal to the dimension k.

($\Rightarrow$ $N((A-\lambda I)^p) = \mathbb{C}^k$)

(4) If $\{\vec{e}_1, \dots, \vec{e}_k\}$ is the standard basis for $\mathbb{C}^k$,

then $(A - \lambda I)^i \vec{e}_i = 0$ for each $i = 1, 2, \dots, k$.

Main Theorem: (Jordan Decomposition Theorem)

Let $A \in M_{n\times n}(\mathbb{C})$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (distinct) with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Then:

(1) $\dim(K_{\lambda_i}) = m_i$

(2) $\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$

(3) Each K_{λ_i} has a basis $\beta_i = \gamma_{1,i} \cup \cdots \cup \gamma_{\ell,i}$ where every $\gamma_{m,i}$ is a cycle:

$$\gamma_{m,i} = \left\{ (A-\lambda_i I)^{p-1}\vec{x}, (A-\lambda_i I)^{p-2}\vec{x}, \ldots, (A-\lambda_i I)\vec{x}, \vec{x} \right\}$$

Consider $\gamma = \{ \underbrace{(A-\lambda_i I)^{p-1}\vec{x}}_{\vec{w}_1}, \underbrace{(A-\lambda_i I)^{p-2}\vec{x}}_{\vec{w}_2}, \ldots, \underbrace{\vec{x}}_{\vec{w}_p} \}$

$$\vec{w}_1 = (A-\lambda_i I)\vec{w}_2 \Rightarrow A\vec{w}_2 = \vec{w}_1 + \lambda_i \vec{w}_2$$

$$[L_A]_\gamma = \begin{pmatrix} | & | & \\ [A\vec{w}_1]_\gamma & [A\vec{w}_2]_\gamma & \cdots \\ | & | & \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_i & 1 & 0 & & O \\ 0 & \lambda_i & 1 & & \\ 0 & 0 & \lambda_i & \ddots & \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & \cdots & \vdots & & \lambda_i \end{pmatrix}$$

a Jordan block.

Example: $A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$.

Step 1: Eigenvalues $\lambda = -1$, alg. mult. $= 3$

Step 2: Eigenspace $E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$\Rightarrow \dim(E_{-1}) = 1 < 3$

Step 3: Jordan Canonical Form

$$J = \begin{pmatrix} -1 & 1 & 0 \\ & -1 & 1 \\ 0 & & -1 \end{pmatrix}$$

Step 4: Find the basis for K_{-1}

$$\beta = \{ (A+I)^2 \vec{v}, \ (A+I)\vec{v}, \ \vec{v} \}$$

Find $\vec{v} \in N((A+I)^3)$ but $\vec{v} \notin N((A+I))$

$\vec{v} \notin N((A+I)^2)$

$$N(A+I) = \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$$

$$N((A+I)^2) = \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

Take $\vec{v} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Then: $\beta = \left\{\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

$$\Rightarrow Q^{-1}AQ = \begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{pmatrix} \text{ whee } Q = \begin{pmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

<u>Theoretical proof:</u> $T: V \to V$ (fin-dim), char poly splits.
($\lambda_1, \lambda_2, \dots, \lambda_k$ distinct eigenvalues)

<u>Claim 1:</u> $(T - \lambda_i I)\big|_{K_{\lambda_j}} : K_{\lambda_j} \to K_{\lambda_j}$ is 1-1 if $i \neq j$.

<u>Pf:</u> Let $\vec{x} \in K_{\lambda_j}$ and $(T - \lambda_i I)(\vec{x}) = \vec{0}$.

Let $p =$ smallest integer $\ni (T - \lambda_j I)^p (\vec{x}) = \vec{0}$.

Let $\vec{y} = (T - \lambda_j I)^{p-1}(\vec{x}) \neq \vec{0}$. Then: $(T - \lambda_j I)(\vec{y}) = \vec{0}$

$\therefore \vec{y} \in E_{\lambda_j}$

Also, $(T - \lambda_i I)(\vec{y}) = (T - \lambda_i I)(T - \lambda_j I)^{p-1}(\vec{x})$

$$= (T - \lambda_j I)^{p-1} \underbrace{(T - \lambda_i I)(\vec{x})}_{\vec{0}} = \vec{0}$$

$\therefore \vec{y} \in E_{\lambda_i}$

$\Rightarrow \vec{y} \in E_{\lambda_i} \cap E_{\lambda_j} = \{\vec{0}\} \Rightarrow \vec{y} = \vec{0}$ (Contradiction)

$N((T - \lambda_j I|_{K_{\lambda_j}}) = \{\vec{0}\} \Rightarrow (T - \lambda_j I)\big|_{K_{\lambda_j}}$ is 1-1.

Claim 2: $\dim(K_{\lambda_i}) \leq m_i$ = multiplicity of λ_i and

$$K_{\lambda_i} = N\left((T - \lambda_i I)^{m_i}\right)$$

Pf:

(1) Let $g(t)$ = char poly of $T|_{K_{\lambda_i}}$

Then: $g(t) \mid$ char poly of T.

Now, $(T - \lambda_j I)|_{K_{\lambda_i}}(\vec{x}) \neq \vec{0}$ if $\lambda_i \neq \lambda_j$ and $\vec{x} \neq \vec{0}$

$\therefore$ λ_i is the ONLY eigenvalue of $T|_{K_{\lambda_i}}$

$\therefore$ $g(t) = (\lambda_i - t)^d$, $d = \dim(K_{\lambda_i})$

$\therefore$ $d \leq m_i$

(2) $N((T-\lambda_i I)^{m_i}) \subseteq K_{\lambda_i}$ (by definition)

Now, by Caley - Hamilton Thm,

$$g(T|_{K_{\lambda_i}}) = 0$$

Char poly of $T|_{K_{\lambda_i}}$

$$\therefore \quad (T|_{K_{\lambda_i}} - \lambda_i I)^d = 0 \underset{(d \le m_i)}{\Rightarrow} (T-\lambda_i I)^d(\vec{x}) = \vec{0} \text{ for } \forall \vec{x} \in K_{\lambda_i}$$

$$\Rightarrow (T-\lambda_i I)^{m_i}(\vec{x}) = \vec{0} \text{ for } \forall \vec{x} \in K_{\lambda_i}$$

$$\therefore \quad K_{\lambda_i} \subseteq N((T-\lambda_i I)^{m_i})$$

<u>Claim 3:</u> $V = K_{\lambda_1} + K_{\lambda_2} + \dots + K_{\lambda_k}$

<u>Pf:</u> By M.I. on k = # of distinct eigenvalues.

When $k=1$, let m = multiplicity of λ_1. Then, char poly of T "$(\lambda_1 - t)^m$".

By Caley-Hamilton Thm, $g(T) = (\lambda_1 I - T)^m = 0 \leftarrow$ zero transf.

$\therefore \quad K_{\lambda_1} = N((T - \lambda_1 I)^m) = V$

$\therefore$ Thm is true for $k=1$.

Assume that the thm is true for any lin. op. w/ fewer than k distinct eigenvalues.

Consider $T: V \to V$ with k distinct eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_k$

Claim 4: Let $W = R\left((T - \lambda_k I)^{m_k}\right)$ ← multiplicity of λ_k

Then: ① $T|_W : W \to W$ is well-defined. (exercise)

② $T|_W$ has $k-1$ distinct eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$

③ $(T - \lambda_k I)^{m_k}\big|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$ is onto $(i < k)$