Lecture 21:

Jordan Canonical Form

Recall: Let $T: V \to V$ lin. operator on a fin-dim V (over F)

$T: V \to V$ is diagonalizable $\iff$

① Char poly splits.

② $\dim(E_{\lambda_i}) = m_i \leftarrow$ alg. multiplicity for all eigenvalues λ_i

(In general, $\dim(E_{\lambda_i}) \leq m_i$)

Remark: "Diagonalizable" (as a linear transf) $\iff$ eigenspaces are BIG enough

Examples:

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \Rightarrow \begin{cases} ① \ 1 \text{ eigenvalue} \\ ② \ \dim(E_\lambda) = 1 \end{cases}$$

$$B = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow \begin{cases} ① \ 1 \text{ eigenvalue} \\ ② \ \dim(E_\lambda) = 1 \end{cases}$$

$$K = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \Rightarrow \begin{cases} ① \ 1 \text{ eigenvalue} \\ ② \ \dim(E_\lambda) = 1 \end{cases}$$

Theorem: Any $A \in M_{n\times n}(\mathbb{C})$ is similar to a matrix of the following form:

$$J = \begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & & 0\\ & \lambda_1 & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_1\end{matrix}} & & & \\ & \boxed{\begin{matrix}\lambda_2 & 1 & & 0\\ & \lambda_2 & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_2\end{matrix}} & & \text{\Large 0} \\ & & \ddots & \\ \text{\Large 0} & & & \boxed{\begin{matrix}\lambda_N & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_N\end{matrix}}\end{pmatrix}$$

(Jordan Canonical Form of A)

$\lambda_1, \lambda_2, \ldots, \lambda_N$ are eigenvalues of A (not necessarily distinct)

$$\begin{pmatrix} \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} & & \\ & \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} & \\ & & \begin{pmatrix} 3 & 1 \\ & 3 \end{pmatrix} \end{pmatrix}$$

Remark: Jordan canonical consists of blocks in this form:

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \lambda & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in M_{k\times k}(\mathbb{C})$$

It is called Jordan block of size k with eigenvalue λ.

Prop: (1) A has only 1 eigenvalue λ (multiplicity is k)

(2) $\dim(E_\lambda) = 1$ ($\Rightarrow$ A is not diagonalizable if $k \neq 1$)

(3). The smallest positive integer p s.t.

$(A - \lambda I)^p = 0$ is equal to the dimension k.

($\Rightarrow$ $N((A - \lambda I)^p) = \mathbb{C}^k$)

(4) If $\{\vec{e}_1, \ldots, \vec{e}_k\}$ is the standard basis for $\mathbb{C}^k$,

then $(A - \lambda I)^i \vec{e}_i = 0$ for each $i = 1, 2, \ldots, k$.

e.g. $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$. Then: $(A - \lambda I) = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$

$$(A - \lambda I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

$$(A - \lambda I)^3 = 0$$

<u>Definition:</u> Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{C})$. $\vec{x} \in \mathbb{C}^n$ is a generalized eigenvector of A corresponding to the eigenvalue λ if (i) $\vec{x} \neq \vec{0}$

and (ii) $(A - \lambda I)^p \vec{x} = 0$ for some positive integer p.

We denote the generalized eigenspace by:

$$K_\lambda = \{ \vec{x} \in \mathbb{C}^n : (A - \lambda I)^p \vec{x} = \vec{0} \text{ for some } p \geq 1 \}$$

Main Theorem: (Jordan Decomposition Theorem)

Let $A \in M_{n \times n}(\mathbb{C})$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (distinct) with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Then:

(1) $\dim(K_{\lambda_i}) = m_i$

(2) $\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$

(3) Each K_{λ_i} has a basis $\beta_i = \gamma_{1,i} \cup \cdots \cup \gamma_{\ell,i}$ where every $\gamma_{m,i}$ is a cycle:

$$\gamma_{m,i} = \{(A-\lambda_i I)^{p-1}\vec{x}_m, (A-\lambda_i I)^{p-2}\vec{x}_m, \ldots, (A-\lambda_i I)\vec{x}_m, \vec{x}_m\}$$

where: $(A-\lambda_i I)^p \vec{x}_m = \vec{0}$

eigenvector

gives rise to $\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \lambda_i & \ddots \\ 0 & & & \lambda_i \end{pmatrix}$

Remark: For a Jordan block, $K_\lambda = \mathbb{C}^k$.

$$\underbrace{\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}}_{k} \Bigg\} k$$

Consider $\gamma = \{ \underbrace{(A-\lambda_i I)^{p-1}\vec{x}}_{=\vec{w}_1}, \underbrace{(A-\lambda_i I)^{p-2}\vec{x}}_{=\vec{w}_2}, \dots, \underbrace{\vec{x}}_{=\vec{w}_p} \}$

$$\vec{w}_1 = (A-\lambda_i I)\vec{w}_2 \Rightarrow A\vec{w}_2 = \vec{w}_1 + \lambda_i \vec{w}_2 \quad \dots$$

$$[L_A]_\gamma = \begin{pmatrix} \Big| & \Big| & \\ [A\vec{w}_1]_\gamma & [A\vec{w}_2]_\gamma & \cdots \\ \Big| & \Big| & \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_i & 1 & 0 & & \\ 0 & \lambda_i & 1 & & O \\ 0 & 0 & \lambda_i & \ddots & \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & \cdots & \vdots & & \lambda_i \end{pmatrix}$$

& Jordan block.

Example: $A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$

Step 1: Compute eigenvalues.

$f(t) = -(t-1)^3 \Rightarrow$ ONLY 1 eigenvalue $\lambda = 1$, alg. mult $= 3$.

Step 2: Find eigenspace

$$E_1 = N(A - 1I) = N\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$\dim(E_1) = 2 < 3$ ($\Rightarrow A$ is NOT diagonalizable)

Step 3: Decide the Jordan Canonical Form

$$\cancel{J = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{matrix}} \end{pmatrix}} \quad \text{or} \quad \cancel{J = \begin{pmatrix} \boxed{1} & & \\ & \boxed{1} & \\ & & \boxed{1} \end{pmatrix}} \quad \text{or} \quad J = \begin{pmatrix} \boxed{1} & & \\ & \boxed{\begin{matrix} 1 & 1 \\ & 1 \end{matrix}} \end{pmatrix}$$

Step 4: Find basis K_λ consisting of cycles.

$$\beta = \gamma_1 \cup \gamma_2 = \{\vec{v}_1\} \cup \{\underbrace{(A - \underset{\overset{\shortparallel}{1}}{\lambda} I)\vec{v}_2, \vec{v}_2}_{\gamma_2}\}$$

eigenvectors

Need $\vec{v}_2 \in N((A-\lambda I)^2)$ but $\vec{v}_2 \notin N(A-\lambda I) = E_1$

$$(A - 1I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Then: $\gamma_2 = \{(A-\lambda I)\vec{v}_2, \vec{v}_2\} = \left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

Take $\vec{v}_1 \in E_1 \Rightarrow \vec{v}_1$ is not parallel to $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Choose $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad Q^{-1} A Q = \begin{pmatrix} \boxed{1} & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} \quad \text{where} \quad Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$