Lecture 20:

Spectral Decomposition:

Prop: Let V be an inner product space and $W \subset V$ a fin-dim subspace with an o.n. basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$. Then: the orthogonal projection $T: V \to V$ defined by:

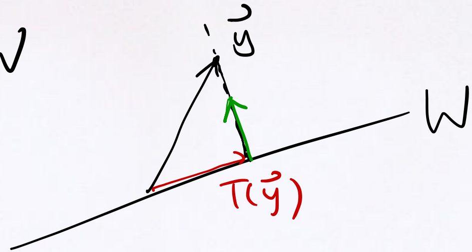
$$T(\vec{y}) = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

is a linear operator s.t.

(1) $N(T) = W^{\perp}$ and $R(T) = W$

(2) $T^2 = T$

(3) T is self-adjoint.



(Orthogonal projection $\iff$ $R(T) = N(T)^{\perp}$ and $R(T)^{\perp} = N(T)$)

Pf: T is linear because $\langle \cdot, \cdot \rangle$ is linear in the first argument

$$N(T) = \left\{ \vec{y} \in V : \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i = \vec{0} \right\}$$

$$= \left\{ \vec{y} \in V : \langle \vec{y}, \vec{v}_i \rangle = 0 \text{ for } i = 1, 2, \ldots, k \right\} = W^{\perp}$$

By definition, $R(T) \subset W$

For $\forall \vec{u} \in W$, we have: $\vec{u} = \sum_{i=1}^{k} \langle \vec{u}, \vec{v}_i \rangle \vec{v}_i = T(\vec{u}) \in R(T)$

$\therefore$ $W = R(T)$ and $T|_W = I_W$

$$\therefore \quad T^2 = T \circ T = T|_{\underset{W}{\overset{\shortparallel}{R(T)}}} \circ T = I_W \circ T = T$$

For any $\vec{x}, \vec{y} \in V$, write $\vec{x} = \vec{x}_1 + \vec{x}_2$ $\quad \vec{x}_1 \in W$, $\vec{x}_2 \in W^{\perp}$

$\vec{y} = \vec{y}_1 + \vec{y}_2$ $\quad \vec{y}_1 \in W$, $\vec{y}_2 \in W^{\perp}$

Then:

$$\langle \vec{x}, T(\vec{y}) \rangle = \langle \underset{\substack{\uparrow \\ W}}{\vec{x}_1} + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{x}_2}, \underset{\substack{\| \\ \vec{y}_1 \\ \uparrow \\ W}}{T(\vec{y}_1)} + \underset{\substack{\| \\ \vec{0}}}{T(\vec{y}_2)} \rangle = \langle \vec{x}_1, \vec{y}_1 \rangle$$

$$\langle T(\vec{x}), \vec{y} \rangle = \langle \underset{\substack{\| \\ \vec{x}_1 \\ \uparrow \\ W}}{T(\vec{x}_1)} + \underset{\substack{\| \\ \vec{0}}}{T(\vec{x}_2)}, \underset{\substack{\uparrow \\ W}}{\vec{y}_1} + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{y}_2} \rangle = \langle \vec{x}_1, \vec{y}_1 \rangle$$

$$\| $$

$$\langle \vec{x}, T^{*}(\vec{y}) \rangle$$

$\therefore \quad T^{*} = T \Rightarrow T$ is self-adjoint.

Thm: Let T be a linear operator on a fin-dim inner product space V over F with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (spectrum of T)

Assume T is normal (resp. self-adjoint) if $F = \mathbb{C}$ (resp $F = \mathbb{R}$)

For $i = 1, 2, \ldots, k$, let $E_i = E_{\lambda_i} = \{\vec{x} \in V : T(\vec{x}) = \lambda_i \vec{x}\}$.

and let T_i be the orthogonal projection onto E_i.

Then:

(a) $V = E_1 \oplus E_2 \oplus \ldots \oplus E_k$

(b) $E_i^{\perp} = \bigoplus_{j \neq i} E_j$ for $i = 1, 2, \ldots, k$

(c) $T_i T_j = \delta_{ij} T_j$ for $1 \leq i, j \leq k$

(d) $I = T_1 + T_2 + \ldots + T_k$ ← Resolution of the identity transformation.

(e) $T = \lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_k T_k$ ← Spectral decomposition.

Remark: $V = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ means -

① $V = E_1 + E_2 + \cdots + E_k \stackrel{\text{def}}{=} \{\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k : \vec{x}_j \in E_j \text{ for } j = 1, 2, \ldots, k\}$

② $E_i \cap \left(\sum_{j \neq i} E_j \right) = \{\vec{0}\}$ for $\forall i$

Consequences: ① $\dim(V) = \dim(E_1) + \cdots + \dim(E_k)$

② For any $\vec{v} \in V$,

$\vec{v}$ can be written uniquely as

$$\vec{v} = \underset{\underset{E_1}{\wedge}}{\vec{x}_1} + \cdots + \underset{\underset{E_k}{\wedge}}{\vec{x}_k}$$

Pf: (a) This follows from the fact that T is diagonalizable

$\exists$ o.n. basis of eigenvectors $\beta = \{ \underbrace{\vec{v}_1, \vec{v}_2}_{E_1}, \underbrace{\ldots, \vec{v}_i}_{E_2}, \ldots, \underbrace{\vec{v}_n}_{E_k} \}$

(b) $\because$ $E_j \subset E_i^{\perp}$ for $j \neq i$ $\therefore$ $\bigoplus_{j \neq i} E_j \subset E_i^{\perp}$

Now, $\dim(E_i^{\perp}) = \dim(V) - \dim(E_i)$

$$= \sum_{j \neq i} \dim(E_j) = \dim\Big(\bigoplus_{j \neq i} E_j\Big)$$

$\therefore$ $E_i^{\perp} = \bigoplus_{j \neq i} E_j$

(c) $T_i T_j = T_i\big|_{R(T_j)} T_j = \delta_{ij} I\big|_{E_j} T_j = \delta_{ij} T_j$

$R(T_j) = E_j \subset E_i^{\perp}$ if $j \neq i$

$$= \begin{cases} I|_{E_j} \circ T_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(d) + (e) : $\because$ $V = E_1 \oplus E_2 \oplus \cdots \oplus E_k$

$\therefore$ for any $\vec{x} \in V$, $\vec{x}$ can be written uniquely as:

$$\vec{x} = \vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k \,, \quad \vec{x}_i \in E_i \text{ for } \forall\, i = 1, 2, \ldots, k.$$

Then: $T_i(\vec{x}) = T_i(\vec{x}_1 + \cdots + \vec{x}_k) = \vec{x}_i$

$$\Rightarrow \quad (T_1 + T_2 + \cdots + T_k)(\vec{x}) = \vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k = \vec{x} \quad \forall\, \vec{x}$$

$$\therefore \quad T_1 + T_2 + \cdots + T_k = I$$

Also,
$$\begin{aligned} T(\vec{x}) &= T(\vec{x}_1) + T(\vec{x}_2) + \cdots + T(\vec{x}_k) \\ &= \lambda_1 \underset{\overset{\parallel}{T_1(\vec{x})}}{\vec{x}_1} + \lambda_2 \underset{\overset{\parallel}{T_2(\vec{x})}}{\vec{x}_2} + \cdots + \lambda_k \underset{\overset{\parallel}{T_k(\vec{x})}}{\vec{x}_k} \\ &= (\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k)(\vec{x}) \end{aligned}$$

$$\therefore \quad T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k.$$

<u>Cor</u>: If $F = \mathbb{C}$, then T is normal iff $T^* = g(T)$ for some polynomial g.

<u>Pf</u>: ($\Rightarrow$) Suppose T is normal. Let $T = \lambda_1 T_1 + \dots + \lambda_k T_k$ be spectral decomposition of T.

Then:
$$T^* = \overline{\lambda_1} T_1^* + \overline{\lambda_2} T_2^* + \dots + \overline{\lambda_k} T_k^*$$
$$= \overline{\lambda_1} T_1 + \overline{\lambda_2} T_2 + \dots + \overline{\lambda_k} T_k$$

By Lagrange interpolation, $\exists$ a polynomial g s.t. $g(\lambda_i) = \overline{\lambda_i}$ $\forall i = 1, 2, \dots k$

Then:
$$g(T) = g(\lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k)$$
$$= g(\lambda_1) T_1 + g(\lambda_2) T_2 + \dots + g(\lambda_k) T_k \quad \text{(Check)}$$
$$= \overline{\lambda_1} T_1 + \overline{\lambda_2} T_2 + \dots + \overline{\lambda_k} T_k = T^*$$

$g = x^2 + x$

$(\lambda_1 T_1 + \lambda_2 T_2)^2 + (\lambda_1 T_1 + \lambda_2 T_2)$

$\lambda_1^2 T_1^2 + \lambda_2^2 T_2^2$, $T_2^2 = T_2$, $T_1^2 = T_1$

$+ \lambda_1 \lambda_2 \cancel{T_1 T_2}^{0} + \lambda_1 \lambda_2 \cancel{T_2 T_1}^{0}$

$(\Leftarrow)$ If $T^* = g(T)$, then: $T^*T = g(T)\,T = T\,g(T)$

$= T\,T^*$

$\therefore$ T is normal.