Lecture 19:

Def: Let T be a linear operator on finite-dim inner product space V over F. If $\|T(\vec{x})\| = \|\vec{x}\|$ $\forall \vec{x} \in V$, then we call T is a unitary linear operator. (resp. orthogonal operator) if $F = \mathbb{C}$ (resp $F = \mathbb{R}$)

Lemma: Let U be a self-adjoint linear operator on a fin-dim inner product space V. If $\langle \vec{x}, U(\vec{x}) \rangle = 0$ $\forall \vec{x} \in V$, then $U = T_0 =$ zero transf.

Pf: Choose an orthonormal basis β for V consisting of eigenvectors of U.

If $\vec{x} \in \beta$, then $U(\vec{x}) = \lambda \vec{x}$ for some λ.

$$0 = \langle \vec{x}, U(\vec{x}) \rangle = \langle \vec{x}, \lambda \vec{x} \rangle = \bar{\lambda} \langle \vec{x}, \vec{x} \rangle = \bar{\lambda} \|\vec{x}\|^2$$

$$\Rightarrow \quad \lambda = 0$$

$\therefore \quad U(\vec{x}) = 0$ for $\forall \vec{x} \in \beta$

$\therefore \quad U = T_0$.

<u>Thm:</u> For a linear operator T on a fin-dim inner product space V, the following are equivalent:

(a) $TT^* = T^*T = I$

(b) T preserves the inner product on V, i.e.,

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle \quad \forall \vec{x}, \vec{y} \in V.$$

(c) $T(\beta) \stackrel{\text{def}}{=} \{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ is an orthonormal basis for V for any orthonormal basis β for V

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

(d) $\exists$ an orthonormal basis β for V s.t. $T(\beta)$ is an orthonormal basis for V.

(e) $\|T(\vec{x})\| = \|\vec{x}\|$ for $\forall \vec{x} \in V$

Proof: (a) $\Rightarrow$ (b) : $\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \underset{\overset{\shortparallel}{I}}{T^*} T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$

(b) $\Rightarrow$ (c) : Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be an orthonormal basis for V

Then $\langle T(\vec{v}_i), T(\vec{v}_j) \rangle = \langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore$ $T(\beta)$ is an orthonormal basis for V.

(c) $\Rightarrow$ (d) : Obvious

(d) $\Rightarrow$ (e) : Let $\vec{x} \in V$, and $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ = o.n. basis for V.

$\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i$ for some $a_1, \ldots, a_n \in F$. $\Rightarrow \|\vec{x}\|^2 \overset{\text{def}}{=} \langle \vec{x}, \vec{x} \rangle = \sum_{i=1}^{n} |a_i|^2$

(Check)

But $T(\beta)$ is o.n. basis.

$\therefore \|T(\vec{x})\|^2 = \left\| \sum_{i=1}^{n} a_i T(\vec{v}_i) \right\|^2 = \sum_{i=1}^{n} |a_i|^2$ $\quad \therefore \|T(\vec{x})\| = \|\vec{x}\|$

for $\forall \vec{x} \in V$.

$(e) \Rightarrow (a)$: $\forall \vec{x} \in V$, $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2 = \|T(\vec{x})\|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle$

$= \langle \vec{x}, T^* T(\vec{x}) \rangle$

$$\Rightarrow \langle \vec{x}, \overbrace{(I - T^* T)}^{U}(\vec{x}) \rangle = 0 \quad \text{for all } \vec{x} \in V.$$

$\underset{\text{Self-adjoint.}}{\parallel}$

By lemma, we know $I - T^* T = T_0 \Rightarrow T^* T = I$

Similarly, we can show $T T^* = I$

Def: A matrix $A \in M_{n\times n}(\mathbb{R})$ is called orthogonal if:

$$A^T A = AA^T = I$$

The set of orthogonal real matrices is denoted as $O(n)$

A matrix $A \in M_{n\times n}(\mathbb{C})$ is called unitary if:

$$A^* A = AA^* = I$$

The set of unitary complex matrices is denoted as $U(n)$

Remark:

- T is unitary (or orthogonal) iff $\exists$ an o.n. basis β s.t. $[T]_\beta$ is unitary (resp. orthogonal).
  $\left([T^*]_\beta = ([T]_\beta)^*\right)$
- Let $\vec{v}_1, \dots, \vec{v}_n \in F^n$. Then: $A \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}(F)$ is unitary (or orthogonal) iff $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an o.n. basis for $\mathbb{C}^n$ (resp $\mathbb{R}^n$)

Thm: Let $A \in M_{n\times n}(\mathbb{C})$. Then: L_A is normal iff A is unitarily equivalent to a diagonal matrix.

(That is, $\exists\, P \in U(n)$ s.t. $P^* A P$ is diagonal)

Pf: ($\Rightarrow$) Suppose L_A is normal. Then: $\exists$ an o.n. basis β, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of eigenvectors for $\mathbb{C}^n$ s.t. $[L_A]_\beta = P^{-1} \underset{\underset{[L_A]_{\beta'}}{\parallel}}{A} P$ is diagonal, where $P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$

$\beta' =$ standard ordered basis

$\therefore$ P is unitary, $P^* P = P P^* = I \Rightarrow P^{-1} = P^*$.

($\Leftarrow$) Obvious. Exercise.

Thm: Let $A \in M_{n \times n}(\mathbb{R})$. Then: A is symmetric iff A is orthogonally equivalent to a diagonal matrix.

That is, $\exists P \in O(n)$ s.t. $P^T A P$ is diagonal.

e.g.

Consider $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$. Then $\exists\, P \in O(3)$ s.t. $P^t A P$ is diagonal.

To find P explicitly, we first compute the eigenvalues of A:

$$f_A(t) = (8-t)(2-t)^2$$

So the eigenvalues are $\lambda = 2$ and $\lambda = 8$

For $\lambda = 8$, $(1,1,1)$ is an eigenvector

For $\lambda = 2$, $\{(-1,1,0), (-1,0,1)\}$ is a basis for the eigenspace E_2 but it is not orthogonal.

Applying the Gram-Schmidt process produces the orthogonal basis $\{(-1,1,0), (1,1,-2)\}$ of E_2.

Then an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A is given by

$$\left\{ \frac{1}{\sqrt{2}}(-1,1,0),\ \frac{1}{\sqrt{6}}(1,1,-2),\ \frac{1}{\sqrt{3}}(1,1,1) \right\}$$

which gives P as

$$P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2\sqrt{6} & 1/\sqrt{3} \end{pmatrix}.$$

Spectral Decomposition:

Prop: Let V be an inner product space and $W \subset V$ a fin-dim subspace with an o.n. basis $\{\vec{v}_1, \dots, \vec{v}_k\}$. Then: the orthogonal projection $T: V \to V$ defined by:

$$T(\vec{y}) = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

is a linear operator s.t.

(1) $N(T) = W^{\perp}$ and $R(T) = W$

(2) $T^2 = T$

(3) T is self-adjoint.