Lecture 18:

Recall: Let V be a finite-dim inner product space.

Let T be a linear operator on V.

① The adjoint of T is a linear operator $T^*: V \to V$ such that:

$$\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle \text{ for } \forall \vec{x}, \vec{y} \in V.$$

② Let V be a finite-dim inner product space and let β be an orthonormal basis for V. Then: $\forall T: V \to V$, we have:

$$[T^*]_\beta = \left([T]_\beta\right)^* \leftarrow \text{conjugate tranpose}$$

$$\left(A^* = (\overline{A})^T \right)$$

③ If T has an eigenvector, then so does T^*.

<u>Thm</u> (Schur) Let T be a lin. operator on a finite-dim inner product space. Suppose the char. poly of T splits.

Then: $\exists$ an orthonormal basis β for V s.t. $[T]_\beta$ is upper triangular.

<u>Pf</u>: We prove by induction on $n = \dim(V)$.

The $n=1$ case is obvious.

Assume the statement holds for lin. operators defined on $(n-1)$-dim inner product space, whose char. poly splits

By lemma, T^* has a unit eigenvector $\vec{z}$.

Let $W \overset{\text{def}}{:=} \operatorname{span}\{\vec{z}\}$ and suppose $T^*(\vec{z}) = \lambda \vec{z}$.

<u>Claim:</u> $W^\perp$ is T-invariant.

<u>Pf:</u> Let $\vec{y} \in W^\perp$ and $\vec{x} = c\vec{z} \in W$. Then:

$$\begin{aligned} \langle T(\vec{y}), \vec{x} \rangle = \langle T(\vec{y}), c\vec{z} \rangle &= \langle \vec{y}, c\,T^*(\vec{z}) \rangle \\ &= \langle \vec{y}, c\lambda \vec{z} \rangle \\ &= \bar{c}\bar{\lambda} \langle \underset{\substack{\uparrow \\ W^\perp}}{\vec{y}}, \underset{\substack{\uparrow \\ W}}{\vec{z}} \rangle = 0 \end{aligned}$$

$\therefore$ $T(\vec{y}) \in W^\perp$.

Now, $f_{T_{W^\perp}}(t) \mid f_T(t) \Rightarrow f_{T_{W^\perp}}(t)$ splits. (1)

Also, $\dim(W^\perp) = n-1$ (2)

$\therefore$ Induction hypothesis gives an orthonormal basis γ for $W^\perp$ s.t. $[T_{W^\perp}]_\gamma$ is upper triangular.

Then: $\beta \overset{def}{=} \gamma \cup \{\vec{z}\}$ is orthonormal basis s.t.

$\gamma \subseteq W^{\perp}$ $\quad$ $\{\vec{z}\} \subseteq W$

$$[T]_\beta = \begin{pmatrix} [T_{W^\perp}]_\gamma & \begin{matrix} * \\ * \\ * \\ * \\ \vdots \\ * \end{matrix} \\ 0 \; \cdots \; 0 & * \end{pmatrix}$$

is upper triangular

Assume T is diagonalizable and assume $\exists$ an orthonormal basis β for V s.t. $[T]_\beta$ is diagonal.

Then: $[T^*]_\beta = ([T]_\beta)^*$ is also diagonal

$$\therefore \quad ([T]_\beta)^*([T]_\beta) = ([T]_\beta)([T]_\beta)^*$$

$$[T^*]_\beta [T]_\beta = [T]_\beta [T^*]_\beta$$

$$[T^*T]_\beta = [TT^*]_\beta$$

$$\Rightarrow \quad \boxed{T^*T = TT^*}$$

Definition: Let V be an inner product space. We say that a linear operator $T: V \to V$ is **normal** if $T^*T = TT^*$.

An $n \times n$ real or complex matrix A is called **normal** if

$$A^* A = A A^*$$

Example:

- Unitary (when $F = \mathbb{C}$) or orthogonal (when $F = \mathbb{R}$) if $T^*T = TT^* = I$
- Hermitian (or self-adjoint) if $T^* = T$
- Skew-Hermitian (or anti-self-adjoint) if $T^* = -T$.

Are normal!

Proposition: Let V be an inner product space, and let T be a normal linear operator on V. Then: we have:

(a) $\|T(\vec{x})\| = \|T^*(\vec{x})\| \quad \forall x \in V$

(b) $T - cI$ is normal $\forall c \in F$.

(c) If $T(\vec{x}) = \lambda\vec{x}$, then: $T^*(\vec{x}) = \overline{\lambda}\vec{x}$

(d) If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$, then: $\vec{x}_1$ and $\vec{x}_2$ are orthogonal.

Proof: (a) $\forall \vec{x} \in V$, we have:

$$\|T(\vec{x})\|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle = \langle T^* T(\vec{x}), \vec{x} \rangle$$
$$= \langle T T^*(\vec{x}), \vec{x} \rangle = \langle T^*(\vec{x}), T^*(\vec{x}) \rangle$$
$$= \|T^*(\vec{x})\|^2$$

(b). $$(T - cI)(T - cI)^* = (T - cI)(T^* - \bar{c} I)$$
$$= TT^* - cT^* - \bar{c}T + c\bar{c} I$$
$$= T^* T - cT^* - \bar{c}T + c\bar{c} I$$
$$= (T - cI)^*(T - cI).$$

(c) Suppose $T(\vec{x}) = \lambda \vec{x}$. Let $U = T - \lambda I$. Then, U is normal (by (b)) and $U(\vec{x}) = \vec{0}$. So, by (a),

$$0 = \|U(\vec{x})\| = \|U^*(\vec{x})\| = \|(T^* - \bar{\lambda} I)(\vec{x})\| \iff T^*(\vec{x}) = \bar{\lambda} \vec{x}.$$

(d) By (c), we have:

$$\begin{aligned}
\lambda_1 \langle \vec{x}_1, \vec{x}_2 \rangle &= \langle T(\vec{x}_1), \vec{x}_2 \rangle = \langle \vec{x}_1, T^*(\vec{x}_2) \rangle \\
&= \langle \vec{x}_1, \overline{\lambda_2}\, \vec{x}_2 \rangle \\
&= \lambda_2 \langle \vec{x}_1, \vec{x}_2 \rangle
\end{aligned}$$

$\lambda_1 \neq \lambda_2$

$$\Longleftrightarrow \quad \overbrace{(\lambda_1 - \lambda_2)}^{\neq 0} \langle \vec{x}_1, \vec{x}_2 \rangle = 0$$

$$\Longrightarrow \quad \langle \vec{x}_1, \vec{x}_2 \rangle = 0$$

Theorem: Let T be a linear operator on a finite-dim complex inner product space V. Then, T is normal iff $\exists$ an orthonormal basis for V consisting of eigenvectors of T.

Proof: $(\Leftarrow)$ Obvious.

$(\Rightarrow)$ Suppose T is normal.

By the Fundamental Thm of algebra, $f_T(t)$ splits.

$\therefore$ Schur's Theorem gives us an orthonormal basis

$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ s.t. $[T]_\beta$ is upper triangular.

$[T]_\beta = \left([T(\vec{v}_1)]_\beta \ \cdots \right)$. In particular, $\vec{v}_1$ is an eigenvector of T.

Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}$ are eigenvectors of T and $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$ are their corresponding eigenvalues

We claim that $\vec{v}_k$ is an eigenvector of T (so by induction, all vectors in β are eigenvectors of T)

Now, $T(\vec{v}_j) = \lambda_j \vec{v}_j \Rightarrow T^*(\vec{v}_j) = \bar{\lambda}_j \vec{v}_j$ for $j = 1, 2, \dots, k-1$

$\because$ $A \overset{def}{:=} [T]_\beta$ is upper triangular

$$T(\vec{v}_k) = A_{1k}\vec{v}_1 + A_{2k}\vec{v}_2 + \dots + A_{kk}\vec{v}_k$$

But : $A_{jk} = \langle T(\vec{v}_k), \vec{v}_j \rangle = \langle \vec{v}_k, T^*(\vec{v}_j) \rangle = \langle \vec{v}_k, \bar{\lambda}_j \vec{v}_j \rangle$

$= \lambda_j \underbrace{\langle \vec{v}_k, \vec{v}_j \rangle}_{0}$

$= 0$

for $j = 1, 2, \dots, k-1$. $\therefore T(\vec{v}_k) = A_{kk}\vec{v}_k$

$\therefore \vec{v}_k =$ eigenvector of T.

Example: Let H be the set of continuous complex-valued functions defined on $[0, 2\pi]$ equipped w/ the inner product

$$\langle f, g\rangle \stackrel{\text{def}}{:=} \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\overline{g(t)}\,dt \quad \text{for } f, g \in H.$$

and the orthonormal subset:

$$S = \{\, f_n(t) \stackrel{\text{def}}{:=} e^{int} : n \in \mathbb{Z} \,\} \subset H$$

Let $\overset{\text{inf dim}}{\overset{\shortparallel}{V}} = \operatorname{span}(S)$ and consider the operators T and U on V defined by:

$$T(f) = \underset{= e^{it} f}{f_1 \cdot f}, \qquad U(f) = \underset{\overset{\shortparallel}{e^{-it} f}}{f_{-1} f}$$

$\therefore \quad T(f_n) = f_{n+1}$ and $U(f_n) = f_{n-1} \quad \forall n \in \mathbb{Z}.$

$$e^{it} e^{int} = e^{i(n+1)t}$$

Then: $\langle T(f_m), f_n \rangle = \langle f_{m+1}, f_n \rangle = \delta_{m+1,n} = \begin{cases} 1 & \text{if } m+1=n \\ 0 & \text{otherwise} \end{cases}$

$$= \delta_{m,n-1}$$

$$= \langle f_m, f_{n-1} \rangle$$

$$= \langle f_m, U(f_n) \rangle$$

$$\Rightarrow \quad U = T^*$$

$\therefore \quad TT^* = TU = I = \underbrace{T^*T}_{=\,UT}$. $\therefore$ T is normal.

However, T has no eigenvectors.

If $f \in V$ is an eigenvector of T, say, $T(f) = \lambda f \ (\lambda \in \mathbb{C})$

Then, we write $f = \sum_{i=n}^{m} a_i f_i$, where $a_m \neq 0$

$$\therefore \sum_{i=n}^{m} a_i f_{i+1} = T(f) = \lambda f = \sum_{i=n}^{m} \lambda a_i f_i$$

$$\Rightarrow f_{m+1} = \frac{1}{a_m}\left(\lambda a_n f_n + \sum_{i=n+1}^{m} (\lambda a_i - a_{i-1}) f_i\right)$$

Contradicting the fact that S is linearly independent.

Def: Let T be a linear operator on an inner product space V. We say T is self-adjoint (Hermitian) if $T^* = T$.

An $n \times n$ real or complex matrix A is called self-adjoint (or Hermitian) if $A^* = A$.

Lemma: Let T be a self-adjoint linear operator on a fin-dim inner product space V. Then:

(a) Every eigenvalue of T is real.

(b) Suppose V is real inner product space. Then, the char. poly of T splits over $\mathbb{R}$.

Proof: (a) Suppose $T(\vec{x}) = \lambda \vec{x}$ for $\vec{x} \neq \vec{0}$.

Then: $T^*(\vec{x}) = \bar{\lambda}\vec{x}$ ($\because$ T is normal)

$$\therefore \lambda\vec{x} = T(\vec{x}) = T^*(\vec{x}) = \bar{\lambda}\vec{x}$$

$$\therefore (\lambda - \bar{\lambda})\underbrace{\vec{x}}_{\neq \vec{0}} = \vec{0} \Rightarrow \lambda = \bar{\lambda}. \quad \therefore \lambda \text{ is real.}$$

(b) Let $n = \dim(V)$, β be an orthonormal basis for V and let $A \stackrel{\text{def}}{=} [T]_\beta$

Then: A is self-adjoint. Consider: $L_A : \mathbb{C}^n \to \mathbb{C}^n$

By (a), the eigenvalues of L_A are real.

By Fundamental Thm of Algebra, $f_{L_A}(t)$ splits into factors of the form $t - \lambda$ where λ is an eigenvalue of L_A.

$\because$ λ is real $\therefore$ $f_{L_A}(t)$ splits over $\mathbb{R}$.

But $f_T(t) = f_{L_A}(t)$. So, the result follows.

Theorem: Let T be a linear operator on a fin-dim real inner product space V. Then T is self-adjoint iff $\exists$ orthonormal basis for V consisting of eigenvectors of T.

Proof: ($\Rightarrow$) Suppose T is self-adjoint. By the Lemma, the char poly of T splits over $\mathbb{R}$. By Schur's Theorem, $\exists$ an orthonormal basis β for V s.t. $A \overset{def}{=} [T]_\beta$ is upper triangular. But:

$$A^* = ([T]_\beta)^* = [T^*]_\beta = [T]_\beta = A$$

So, A is both upper triangular and lower triangular.
Hence, A is diagonal.
$\therefore$ β consists of eigenvectors of T.

$(\Leftarrow)$ Suppose $\exists$ orthurnormal basis β for V s.t. $A = [T]_\beta$ is diagonal.

Then: $[T^*]_\beta = ([T]_\beta)^* = A^t = A = [T]_\beta$

$\therefore \quad T^* = T$

$\therefore \quad T$ is self-adjoint.