Lecture 17:

Recall:

Let V be a finite-dim inner product space. Let T be a linear operator on V.

Then: $\exists!$ linear operator $T^*: V \to V$ such that:

$$\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle \text{ for } \forall \vec{x}, \vec{y} \in V.$$

T^* is called the adjoint of T.

Adjoint of a linear operator

Prop: Let V be a finite-dim. inner product space over F.

Then for any linear transformation $g : V \to F$ (linear functional),

$\exists ! \ \vec{y} \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Theorem: Let V be a finite-dim inner product space. Let T be a linear operator on V. Then: $\exists!$ linear operator $T^*: V \to V$ such that: $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ for $\forall \vec{x}, \vec{y} \in V$.

T^* is called the adjoint of T.

Idea: Given any $\vec{y} \in V$, the map $g_{\vec{y}}: V \to F$ defined by:

$g_{\vec{y}}(\vec{x}) = \langle T(\vec{x}), \vec{y} \rangle$ is linear ($\because$ $\langle \cdot, \cdot \rangle$ is linear in the 1^{st} argument) + T is linear

By the previous proposition, $\exists! \, \vec{y}\,' \in V$ such that $\underbrace{\langle T(\vec{x}), \vec{y} \rangle}_{= g_{\vec{y}}(\vec{x})} = \langle \vec{x}, \vec{y}\,' \rangle$ for all $\vec{x} \in V$.

Now, define: $T^*: V \to V$ by $T^*(\vec{y}) = \vec{y}\,'$.

Proof of the Proposition

Prop: Let V be a finite-dim. inner product space over F.
Then for any linear transformation $g : V \to F$ (linear functional),
$\exists !\, \vec{y} \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Proof: Let $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthornormal basis for V.

Set :
$$\vec{y} = \sum_{i=1}^{n} \overline{g(\vec{v}_i)}\, \vec{v}_i \,.$$

We have :
$$\langle \vec{v}_j, \vec{y} \rangle = \sum_{i=1}^{n} g(\vec{v}_i) \langle \vec{v}_j, \vec{v}_i \rangle = g(\vec{v}_j)$$

$$\Rightarrow \quad g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle \text{ for all } \vec{x} \in V$$

If $\exists\ \vec{y}' \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y}' \rangle$ for $\forall\ \vec{x}$.

then, $\langle \vec{x}, \vec{y} \rangle = g(\vec{x}) = \langle \vec{x}, \vec{y}' \rangle$ for $\forall \vec{x}$

$$\Rightarrow \quad \vec{y} = \vec{y}'.$$

Remark:

$$\langle \vec{x}, T(\vec{y}) \rangle = \overline{\langle T(\vec{y}), \vec{x} \rangle} = \overline{\langle \vec{y}, T^*(\vec{x}) \rangle} = \langle T^*(\vec{x}), \vec{y} \rangle$$

Proposition: Let V be a finite-dim inner product space and let β be an orthonormal basis for V. Then $\forall\, T: V \to V$, we have:

$$[T^*]_\beta = \left([T]_\beta\right)^* \leftarrow \text{conjugate tranpose}$$

$$\left(A^* = (\overline{A})^T \right)$$

Proof: Let $A = [T]_\beta$, $B = [T^*]_\beta$ and $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

$$B_{ij} = \langle T^*(\vec{v}_j), \vec{v}_i \rangle = \langle \vec{v}_j, T(\vec{v}_i) \rangle = \overline{\langle T(\vec{v}_i), \vec{v}_j \rangle}$$

$$= \overline{A_{ji}} \quad //$$

Corollary: Let A be an $n \times n$ matrix. Then: $L_{A^*} = (L_A)^*$ ← adjoint

↑ conjugate transpose

Pf: The standard basis β for F^n is orthonormal.

Then: $[L_A]_\beta = A$.

$$\therefore\ [(L_A)^*]_\beta = \left([L_A]_\beta\right)^* = A^* = [L_{A^*}]_\beta \Rightarrow (L_A)^* = L_{A^*}$$

Let $A = [T]_\beta$, $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ = o.n. basis.

Then: $$T(\vec{v}_j) = \sum_{i=1}^{n} A_{ij} \vec{v}_i$$

$$= \sum_{i=1}^{n} \langle T(\vec{v}_j), \vec{v}_i \rangle \vec{v}_i$$

$$\therefore \quad A_{ij} = \langle T(\vec{v}_j), \vec{v}_i \rangle$$

Proposition: Let V be an inner product space. Let $T, U = V \to V$.

Then:

(a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \bar{c}\, T^* \quad \forall c \in F$

(c) $(TU)^* = U^* T^*$

(d) $(T^*)^* = T$

(e) $I^* = I$

Proof: $\forall \vec{x}, \vec{y} \in V$

(a) $\langle \vec{x}, (T+U)^*(\vec{y}) \rangle = \langle (T+U)(\vec{x}), \vec{y} \rangle = \langle T(\vec{x}), \vec{y} \rangle + \langle U(\vec{x}), \vec{y} \rangle$

$= \langle \vec{x}, T^*(\vec{y}) \rangle + \langle \vec{x}, U^*(\vec{y}) \rangle$

$= \langle \vec{x}, (T^* + U^*)(\vec{y}) \rangle$

$\Rightarrow (T+U)^* = T^* + U^*.$

(b) $\langle \vec{x}, (cT)^*(\vec{y}) \rangle = \langle cT(\vec{x}), \vec{y} \rangle$

$= c \langle T(\vec{x}), \vec{y} \rangle$

$= c \langle \vec{x}, T^*(\vec{y}) \rangle = \langle \vec{x}, \bar{c}\, T^*(\vec{y}) \rangle$

$\therefore (cT)^* = \bar{c}\, T^*$

(c) $\langle \vec{x}, (TU)^*(\vec{y}) \rangle = \langle T(U(\vec{x})), \vec{y} \rangle$

$= \langle U(\vec{x}), T^*\vec{y} \rangle$

$= \langle \vec{x}, U^* T^* \vec{y} \rangle$

$\Rightarrow (TU)^* = U^* T^*.$

(d) $\langle \vec{x}, T(\vec{y}) \rangle = \langle T^*(\vec{x}), \vec{y} \rangle = \langle \vec{x}, (T^*)^*(\vec{y}) \rangle$

$\Rightarrow \quad T = T^{**}$.

(e). follows from the definition.

$$\langle \vec{x}, I(\vec{y}) \rangle = \langle I(\vec{x}), \vec{y} \rangle$$
$$\langle \vec{x}, I(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$$

<u>Remark:</u> Let A and B be $n \times n$ matrices. Then:

(a) $(A+B)^* = A^* + B^*$

(b) $(cA)^* = \bar{c} A^*$

(c) $(AB)^* = B^* A^*$

(d) $A^{**} = A$

(e) $I^* = I$.

Lemma: Let $T: V \to V$ be a linear operator on a finite-dim inner product space V. If T has an eigenvector, then so does T^*.

Pf: Suppose $\vec{v} \in V \setminus \{\vec{0}\}$ is an eigenvector of T with eigenvalue λ.

Then: $\forall \vec{x} \in V$, we have:

$$0 = \langle \vec{0}, \vec{x} \rangle = \langle (T - \lambda I)(\vec{v}), \vec{x} \rangle = \langle \vec{v}, \underbrace{\overset{\overset{T^* - \bar{\lambda} I}{\parallel}}{(T - \lambda I)^*}(\vec{x})}_{\in R(T^* - \bar{\lambda} I)} \rangle$$

$\Rightarrow \vec{v} \in R(T^* - \bar{\lambda} I)^{\perp}$. So, $\dim(R(T^* - \bar{\lambda} I)) < \dim(V)$.

$(\dim(W) + \dim(W^{\perp}) = \dim(V))$

$\Rightarrow \dim(N(T^* - \bar{\lambda} I)) > 0$ $\therefore$ T^* has an eigenvector with eigenvalue $\bar{\lambda}$.