MATH2048: Honours Linear Algebra II

2024/25 Term 1

Homework 9

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-11-25 (Monday) 23:59****.*

1. Let T and U be self-adjoint linear operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.

   Solution. TU is self-adjoint if and only if for any x and y, $\langle TUx, y\rangle = \langle x, TUy\rangle$. Since T and U are self-adjoint, $\langle x, TUy\rangle = \langle Tx, Uy\rangle = \langle UTx, y\rangle$. The result then follows as x and y are arbitrary.

2. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.

   (a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.

   Solution. A number is real if and only if it is equal to its conjugate. So we simply compute $\overline{\langle T(x), x\rangle} = \langle x, T(x)\rangle = \langle T(x), x\rangle$.

   (b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = T_0$.

   Proof. Notice that $\langle Tu, v\rangle = \dfrac{1}{4}(\langle T(u+v), u+v\rangle - \langle T(u-v), u-v\rangle) + \dfrac{i}{4}(\langle T(u+iv), u+iv\rangle - \langle T(u-iv), u-iv\rangle) = 0$ for any u and v. Taking $v = Tu$ for each u, we obtain $Tu = 0$ for any u.

   (c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T^*$.

   Proof. By (b), it suffices to prove $\langle (T - T^*)(x), x\rangle$ for any x. It is obviously true as $\langle T(x), x\rangle = \overline{\langle T(x), x\rangle} = \langle x, T(x)\rangle == \langle T^*(x), x\rangle$ for all x.

3. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$

   $$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

   Proof. $\langle T(x) + ix, T(x) + ix\rangle = \left\|T(x)\right\|^2 + \|x\|^2 + \langle T(x), ix\rangle + \langle ix, T(x)\rangle$. The sum of the last two terms is 0 because T is self-adjoint.

   (a) Deduce that $T - iI$ is invertible and that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

   Proof. Notice that $\left\|(T - iI)x\right\|^2 = \left\|T(x)\right\|^2 + \|x\|^2 \geq \|x\|^2$. So, $(T - iI)x \neq 0$ if $x \neq 0$. This implies $T - iI$ is invertible. So, $(T - iI)(T - iI)^{-1} = I = I^* = ((T - iI)^{-1})^*(T - iI)^* = ((T - iI)^{-1})^*(T + iI)$. This proves $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

(b) Prove that $(T+iI)(T-iI)^{-1}$ is unitary.

Proof. For any x and y, $\langle (T+iI)(T-iI)^{-1}x, (T+iI)(T-iI)^{-1}y\rangle = \langle x, ((T-iI)^{-1})^*(T+iI)^*(T+iI)(T-iI)^{-1}y\rangle = \langle x, (T+iI)^{-1}(T-iI)(T+iI)(T-iI)^{-1}y\rangle = \langle x, y\rangle$. The last equality is due to $(T-iI)(T+iI) = (T+iI)(T-iI)$.

4. Let W be a finite-dimensional subspace of an inner product space V. Define $U : V \to V$ by $U(v_1+v_2) = v_1 - v_2$, where $v_1 \in W$ and $v_2 \in W^\perp$. Prove that U is a self-adjoint unitary operator.

   Proof. Since $\langle U(v_1+v_2), v_1+v_2\rangle = \langle v_1-v_2, v_1+v_2\rangle = \|v_1\|^2 - \|v_2\|^2 = \langle v_1+v_2, U(v_1+v_2)\rangle$, U is a self-adjoint operator. Further, for any $v_1 \in W$, $v_2 \in W^\perp$, $w_1 \in W$, and $w_2 \in W^\perp$, $\langle U(v_1+v_2), U(w_1+w_2)\rangle = \langle v_1-v_2, w_1-w_2\rangle = \langle v_1+v_2, w_1+w_2\rangle$. So, U is also unitary.

5. Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection of V on W, then $I-T$ is the orthogonal projection of V on $W^\perp$.

   Proof For any $w \in W$ and $w^\perp \in W^\perp$, $T(w) = w$ and $T(w^\perp) = 0$. So, $(I-T)(w) = 0$ and $(I-T)(w^\perp) = w^\perp$. Hence, $I-T$ is the orthogonal projection onto $W^\perp$.