MATH2048: Honours Linear Algebra II

2024/25 Term 1

Homework 8

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-11-08 (Friday) 23:59****.*

1. Let $V = \mathbb{C}^3$, $S = \{(1, i, 0), (1 - i, 2, 4i)\} \subset V$.

   (a) Find an orthonormal basis for span(S).

   Proof. Using Gram-Schmidt process, we get $\{\frac{1}{\sqrt{2}}(1, i, 0), \frac{1}{\sqrt{17}}(1 - i, 2, 4i)\}$.

   (b) Extend S to get an orthonormal basis S' of V.

   Proof. We add $\frac{1}{\sqrt{34}}(4i, -4, -i - 1)$.

   (c) Let $x = (3 + i, 4i, -4)$. Prove that $x \in$ span(S).

   Proof. $(3 + i, 4i, -4) = 2(1, i, 0) + i(1 - i, 2, 4i)$.

2. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$.

   Proof. Since W_1 and W_2 are subspaces of $W_1 + W_2$, we have $(W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp$. Conversely, for any $v \in W_1^\perp \cap W_2^\perp$, $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle = 0$ for arbitrary $w_1 \in W_1$ and $w_2 \in W_2$. Hence, $W_1^\perp \cap W_2^\perp \subset (W_1 + W_2)^\perp$. Similar arguments shows the second equality.

3. Let V be the vector space of all sequence σ in F (where $F = \mathbb{R}$ or $\mathbb{C}$) such that $\sigma(n) \neq 0$ for only finitely many positive integers n. For $\sigma, \mu \in V$, we define

$$\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}.$$

   Since all but a finite number of terms of the series are zero, the series converges.

   (a) Prove that $\langle \cdot, \cdot \rangle$ in an inner product on V, and hence V is an inner product space.

   Proof. Conjugate symmetry: $\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)} = \overline{\sum_{n=1}^{\infty} \mu(n)\overline{\sigma(n)}} = \overline{\langle \mu, \sigma \rangle}$. *Linearity*: $\langle a\mu + b\nu, \sigma \rangle = \sum_{n=1}^{\infty}(a\mu(n) + b\nu(n))\overline{\sigma(n)} = a \langle \mu, \sigma \rangle + b \langle \nu, \sigma \rangle$. *Positive-definiteness*: $\langle \mu, \mu \rangle = \sum_{n=1}^{\infty} |\mu(n)|^2 > 0$ for any nonzero μ.

   (b) For each positive integer n, let e_n be the sequence defined by $e_n(k) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta. Prove that $\{e_1, e_2, ...\}$ is an orthonormal basis for V.

   Proof. Trivial arguments.

   (c) Let $\sigma_n = e_1 + e_n$ and $W = \text{span}(\{\sigma_n : n \geq 2\})$.

i. Prove that $e_1 \notin W$, so $W \neq V$.
Proof. Observe that any non-zero element in W contains a non-zero entry for indices larger than 1. The result then follows.

ii. Prove that $W^\perp = \{0\}$, and conclude that $W \neq (W^\perp)^\perp$.
Proof. Pick arbitrary $v \in W^\perp$ and suppose $v = v_1 e_1 + \cdots + v_k e_k + \cdots$. $\langle v, e_1 + e_n \rangle = 0$ implies $v_1 = -v_n$ for any $n > 1$. Besides, $\langle v, e_1 + e_i - (e_1 + e_j) \rangle = 0$ implies $v_j = -v_i$ for any $i, j > 1$ satisfying $i \neq j$. The two equalities force $v_n = 0$ for any $n \geq 1$.

4. Let V and $\{e_1, e_2, ...\}$ be defined as in Q3. Define $T : V \to V$ by

$$T(\sigma)(k) = \sum_{i=k}^{\infty} \sigma(i) \quad \text{for every positive integer } k.$$

Note that the infinite series in the definition of T converges because $\sigma(i) \neq 0$ for only finitely many i.

(a) Prove that T is a linear operator on V.
Proof. Trivial argument.

(b) Prove that for any positive integer n, $T(e_n) = \sum_{i=1}^{n} e_i$.
Proof. Trivial argument.

(c) Prove that T has no adjoint.
Proof. Suppose T has an adjoint T^*. Fix $n \geq 1$. Then, for arbitrary $m \geq n$, $\langle e_m, T^*(e_n) \rangle = \langle T(e_m), e_n \rangle = \langle \sum_{i=1}^{m} e_i, e_n \rangle = 1$. Since $\{e_1, e_2, \cdots\}$ is an orthonormal basis, $\langle e_m, T^*(e_n) \rangle$ is the value of the m-th entry of $T^*(e_n)$. The above results imply that $T^*(e_n)$ is not a sequence with finite non-zero entries, which is a contradiction.

5. Prove that if $V = W \oplus W^\perp$ and T is the projection on W along $W^\perp$, then $T = T^*$.

Proof. It suffices to prove T and T^* coincide on W and $W^\perp$. Then, $T = T^*$ on V follows by linearity of T. Take arbitrary $w \in W$ and $v \in V$ such that $v = w_1 + w_2$ for $w_1 \in W$ and $w_2 \in W^\perp$. So, $\langle T^* w, v \rangle = \langle T^* w, w_1 + w_2 \rangle = \langle w, Tw_1 \rangle + \langle w, Tw_2 \rangle = \langle w, Tw_1 \rangle$ because the projection operator T maps w_2 to 0. We hence have $\langle w - T^* w, w_1 \rangle = 0$ for any $w_1 \in W$, which implies $Tw = w = T^* w$. An almost same argument shows $T^* w^\perp = Tw^\perp$ for any $w^\perp \in W^\perp$.