MATH2048: Honours Linear Algebra II

2024/25 Term 1

Homework 7

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-11-01 (Friday) 23:59****.*

1. Let T be a linear operator on a finite-dimensional vector space V.

   (a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

   Proof. This follows form the fact that $f|_{T_W}(t)$ divides $f|_T(t)$.

   (b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.

   Proof. This directly follows from (a).

2. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-cyclic subspace of V generated by a nonzero vector v. Let $k = \dim(W)$. Prove the following statements.

   (a) $\{x, T(x), T^2(x), ..., T^{k-1}(x)\}$ is a basis for W.

   Proof. Let m be the largest number such that $\{x, T(x), \cdots, T^{m-1}(x)\}$ is linearly independent. Then, we can find nonzero $(a_0, \cdots, a_{m-1})$ such that $T^m(x) = a_0x + \cdots + a_{m-1}T^{m-1}(x)$. Further, $T^{m+1}(x) = T(T^m(x)) = a_0T(x) + \cdots + a_{m-1}T^m(x)$, which is also a linear combination of $\{x, \cdots, T^{m-1}(x)\}$. By induction, $\{x, \cdots, T^{m-1}(x)\}$ generate the T-cyclic subspace W. This implies $m = \dim(W) = k$.

   (b) If $a_0x + a_1T(x) + a_2T^2(x) + ... + a_{k-1}T^{k-1}(x) + T^k(x) = 0$, then $f_{T_W}(t) = (-1)^k(a_0 + a_1t + a_2t^2 + ... + a_{k-1}t^{k-1} + t^k)$.

   Proof. Write $v_0 = v$ and $v_i = T^i(x)$ for $1 \leq i \leq k-1$. $\{v_0, \cdots, v_{k-1}\}$ forms a basis of W. For $0 \leq i \leq k-2$, $T(v_i) = v_{i+1}$, and $T(v_{k-1}) = -(a_0x + a_1T(x) + a_2T^2(x) + ... + a_{k-1}T^{k-1}(x))$. We then have the matrix representation of T with respect to basis $\{v_0, \cdots, v_{k-1}\}$. Some computation gives the formula of $f_{T_W}(t)$.

3. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself (i.e. there exists $x \in V$ such that V is the T-cyclic subspace generated by x). Prove that if U is a linear operator on V, then $UT = TU$ if and only if $U = g(T)$ for some polynomial $g(t)$.

   Proof. The converse direction is trivial. So we only need to prove the forward direction. Suppose $UT = TU$. Let $k = \dim V$. By 2(a), $\{x, T(x), \cdots, T^{k-1}(x)\}$ is a basis of V. Suppose $U(x) = a_0x + \cdots + a_{k-1}T^{k-1}(x)$. By interchanging U and T inductively, we actually have $UT^j = TUT^{j-1} = \cdots = T^jU$ for any $j \geq 1$. So, $U(T^j(x)) = T^j(U(x)) = a_0T^j(x) + \cdots + a_{k-1}T^{k-1+j}(x)$. Hence, $U = a_0T + \cdots + a_{k-1}T^{k-1}$.

4. Let V be an inner product space over F. Prove the following statements.

   (a) If x, y are orthogonal, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.
   Proof. $\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle = \|x\|^2 + \|y\|^2$.

   (b) *Parallelogram law*: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$.
   Proof. $\|x+y\|^2 + \|x-y\|^2 = \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle + \langle x, x \rangle + \langle y, y \rangle - 2\langle x, y \rangle = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$.

   (c) Let $v_1, v_2, ..., v_k$ be an orthogonal set in V, and let $a_1, a_2, ..., a_k \in F$, then

$$\|\sum_{i=1}^{k} a_i v_i\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

   Proof. $\|\sum_{i=1}^{k} a_i v_i\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2 + \sum_{i \neq j} \langle a_i v_i, a_j v_j \rangle = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2$.

5. Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

   Proof. Suppose $T(x) = T(y)$ but $x \neq y$. We instantly see $0 = \|T(x-y)\| = \|x-y\| \neq 0$, which is a contradiction.