

THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

Mathematical Modelling Project Team

mathmodel@math.cuhk.edu.hk

Exercise (Functions)

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Part A: Basic Questions

1. Find the domain of definition of the following functions

(a) $y = \sqrt{4 - x^2} + \frac{1}{x}$.

(b) $y = \sqrt{|x| - 3} + \frac{1}{x^2 - 4x + 5}$.

2. Knowing that the function $y = |x| - x^2$, x takes values in the set

$$\{0, 1, -2, 3, -4, -\frac{1}{2}, \frac{1}{2}, \frac{-1}{3}\}$$

Find the corresponding value of y .

3. It is known that $y = ax^2 + b$ and when $x = 1$, $y = 10$; when $x = \frac{1}{2}$, $y = -8$. Find the value of a and b .
4. In which of the following questions is y a function of x ? Write the expression of the function $y = f(x)$.
- (a) x is the length of one side of the square and y is the area of this square.
- (b) x is the length of one side of the rectangle, the length of the other side of the rectangle is a constant value a , and y is the area of this rectangle.
- (c) One side of the triangle is x , the height on this side is y , and the area of the triangle is a constant S .
5. A car leaves the station and after 45 minutes it arrives at A , 28 kilometers from the station, and thereafter the car has a constant speed of 40 kilometers per hour, find the relationship between the distance s kilometers from the station after this car has a constant speed and the time t hours from the station, and find the domain of t .
6. Knowing that $f(x + 1) = x^2 - 3x + 2$, find $f(x)$.
7. Given that $a < b < c$, find the minimum of the function

$$y = |x - a| + |x - b| + |x - c|$$

8. Find the domain of the following functions.

(a) $y = \frac{2}{x + |x|}$.

(b) $y = \sqrt{x^2 - 3x + 2} + \frac{1}{x^2 + 2x - 8}$.

Part B: Advanced Questions

9. Given the function $f(x)$ satisfying

$$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2},$$

find the value of $f(3)$.

10. Suppose $f(x) = ax^2 + bx + c$, prove that $f(x+3) - 3f(x+2) + 3f(x+1) - f(x) = 0$.

11. Knowing that $f(x)$ satisfying $f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}$, find $f(x)$.

12. Suppose $f(1) = 0$, $f(2) = 1$, and

$$a^2 f(n+2) = b^2 f(n)$$

($a > 0$, $b > 0$, n is a positive integer), find $f(n)$.

13. Let the perimeter of the rectangle be a constant l and the length of one side be x . Find the maximum value of the area y of the rectangle.

14. Suppose $S(t) = \frac{2t}{1+t^2}$, $C(t) = \frac{1-t^2}{1+t^2}$, prove that:

(a) $S(t) = S\left(\frac{1}{t}\right)$.

(b) $S^2(t) + C^2(t) = 1$.

15. Knowing that $f\left(\frac{2x+1}{x}\right) = x^2 - 3x + 7$, find $f(x)$.

16. Suppose $y = ax + \frac{1}{a}(2-x)$, where $a > 0$. Find the minimum of y when $0 \leq x \leq 1$.

Solutions

1. (a) From the question $\begin{cases} 4 - x^2 \geq 0, \\ x \neq 0, \end{cases}$, which means $\begin{cases} -2 \leq x \leq 2, \\ x \neq 0. \end{cases}$

Hence, the domain is $-2 \leq x \leq 2$ and $x \neq 0$.

- (b) From the question $\begin{cases} |x| - 3 \leq 0, \\ x^2 - 4x + 5 \neq 0, \end{cases}$, which means $\begin{cases} x \geq 3 \text{ or } x \leq 3, \\ x \neq 5 \text{ or } x \neq -1. \end{cases}$

Hence, the domain is $(-\infty, -3]$, $[3, 5)$, and $(5, +\infty)$.

2. When $x = 0, y = 0$;

When $x = 1, y = 0$;

When $x = -2, y = -2$;

When $x = 3, y = -6$;

When $x = -4, y = -12$;

When $x = -\frac{1}{2}, y = \frac{1}{4}$;

When $x = \frac{1}{2}, y = \frac{1}{4}$;

When $x = -\frac{1}{3}, y = \frac{2}{9}$.

3. From the question $\begin{cases} a + b = 10, \\ \frac{a}{4} + b = -8. \end{cases}$ Hence, we have $a = 24, b = -14$.

4. (a) $y = x^2$.

(b) $y = xa$.

(c) $y = \frac{2S}{x}$.

5. $s = 28 + 40(t - \frac{3}{4})$, where the domain is $t \geq \frac{3}{4}$.

6. Let $a = x + 1$.

Then, $f(a) = (a - 1)^2 - 3(a - 1) + 2 = a^2 - 5a + 6$.

Substituting a with x gives $f(x) = x^2 - 5x + 6$.

7. We will consider four cases.

(a) $x \leq a$, then $y = a - x + b - x + c - x = a + b + c - 3x \geq b + c - 2a$.

(b) $a \leq x \leq b$, then $y = x - a + b - x + c - x = b + c - a - x \geq c - a$.

(c) $b \leq x \leq c$, then $y = x - a + x - b + c - x = c - a - b + x \geq c - a$.

(d) $c \leq x$, then $y = x - a + x - b + x - c = 3x - a - b - c \geq 2c - a - b$.

Among them, the smallest is $c - a$.

8. (a) From the question, $x + |x| \neq 0$. Hence, $x > 0$ is the domain.

- (b) From the question, $\begin{cases} x^2 - 3x + 2 \geq 0, \\ x^2 + 2x - 8 \neq 0, \end{cases}$. Hence, the domain is $(-\infty, -4)$, $(-4, 1]$, and $(2, +\infty)$.

9. Note that $(x + \frac{1}{x})^2 = x^2 + \frac{1}{x^2} + 2$, we can conclude that $f(y) = y^2 - 2$. Hence, $f(3) = 7$.

10. By direct computation, $f(x+3) - 3f(x+2) + 3f(x+1) - f(x)$
 $= [a(x+3)^2 + b(x+3) + c] - 3[a(x+2)^2 + b(x+2) + c] + 3[a(x+1)^2 + b(x+1) + c] - [ax^2 + bx + c]$
 $= a[(x+3)^2 - 3(x+2)^2 + 3(x+1)^2 - x^2] + b[(x+3) - 3(x+2) + 3(x+1) - x] + c(1 - 3 + 3 - 1)$
 $= 0.$

11. Take $a = \frac{1}{x}$.

Then, $f(a) = \frac{1}{a} + \sqrt{a + \frac{1}{a^2}}.$

Substituting a with x gives $f(x) = \frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}.$

12. We will consider two cases.

(i) When n is an odd number, let $n = 2k + 1$.

Then $f(n) = f(2k+1) = (\frac{b^2}{a^2})f(2(k-1)+1) = (\frac{b^2}{a^2})^2 f(2(k-2)+1) = \dots = (\frac{b^2}{a^2})^k f(1) = 0$

(ii) When n is an even number, let $n = 2k$.

Then $f(n) = f(2k) = (\frac{b^2}{a^2})f(2(k-1)) = (\frac{b^2}{a^2})^2 f(2(k-2)) = \dots = (\frac{b^2}{a^2})^{k-1} f(2) = (\frac{b^2}{a^2})^{k-1}$

13. From the question, $y = (\frac{l}{2} - x)x.$

The minimum value is given by Cauchy inequality, namely $y \leq (\frac{\frac{l}{2} - x + x}{2})^2 = \frac{l^2}{16}.$

14. (a) $S(\frac{1}{t}) = \frac{\frac{2}{t}}{1 + \frac{1}{t^2}} = \frac{2t}{1 + t^2} = S(t).$

(b) $S^2(t) + C^2(t) = \frac{(2t)^2 + (1 - t^2)^2}{(1 + t^2)^2} = \frac{4t^4 + 4t^2 + 1}{4t^4 + 4t^2 + 1} = 1.$

15. Let $y = \frac{2x+1}{x}$, then $x = \frac{1}{y-2}.$

$f(y) = x^2 - 3x + 7 = (\frac{1}{y-2})^2 - \frac{3}{y-2} + 7.$

Substituting y with x gives $f(x) = \frac{1}{(x-2)^2} - \frac{3}{x-2} + 7.$

16. Note that $y = (a - \frac{1}{a})x + \frac{2}{a}$, we will need to consider whether the coefficient of x is positive or not.

(i) If $a - \frac{1}{a} \geq 0$, which means $a \geq 1$, y will reach the minimum when $x = 0$. The minimum is $\frac{2}{a}.$

(ii) If $a - \frac{1}{a} < 0$, which means $0 < a < 1$, y will reach the minimum when $x = 1$. The minimum is $\frac{1}{a} + a.$