On Symmetric Multiple Description Coding

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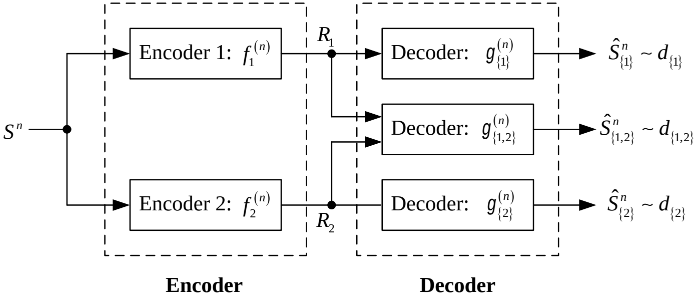
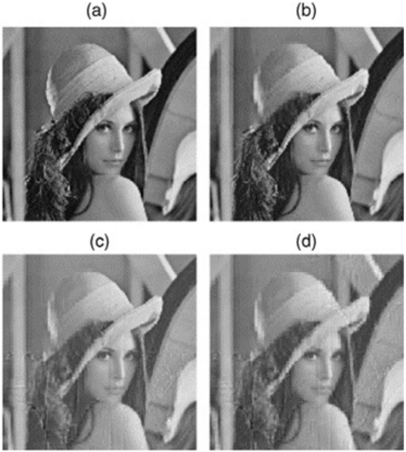
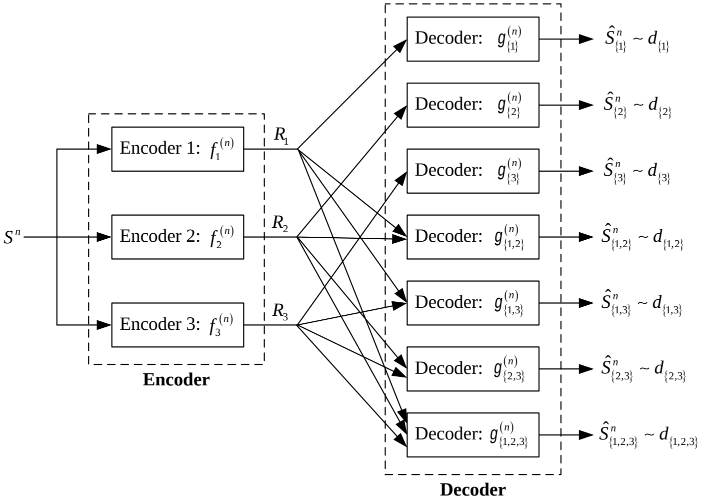
Two Descriptions



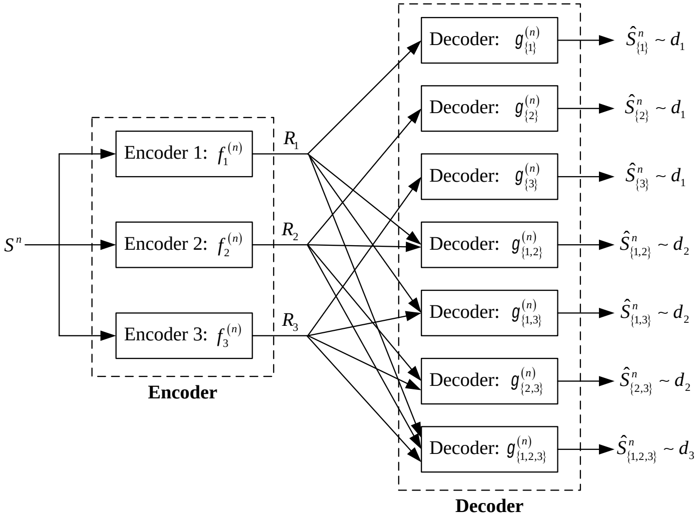


- (a) The original image. (b) The reconstructed image using packets form both encoder 1 and 2. (c) The reconstructed image using packets form encoder 1. (d) The reconstructed image using packets from encoder 2.

Multiple Descriptions

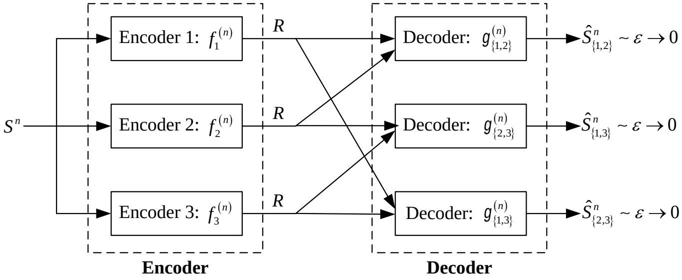


- $R([d])$: the infimum over all achievable sum rates subject to distortion constraints $[d] \triangleq (d_{\mathcal{A}}, \mathcal{A} \in 2_+^{\mathcal{L}})$, $2_+^{\mathcal{A}} = \{\mathcal{B} : \mathcal{B} \subseteq \mathcal{A}, |\mathcal{B}| > 0\}$, $\mathcal{L} = \{1, \cdots, L\}$.



- $R(\underline{d})$: the infimum over all achievable sum rates subject to distortion constraints $\underline{d} \triangleq (d_1, \cdots, d_L)$.

Connection with Distributed Storage



- A single-letter lower bound on the minimum sum rate of multiple description coding with symmetric distortion constraints
- Evaluation of this lower bound in several special cases with the aid of certain minimax theorems
  - the binary uniform source with the erasure distortion measure
  - the binary uniform source with the Hamming distortion measure
  - the quadratic Gaussian case
- A new conclusive result on the information-theoretic limits of Gaussian multiple description coding

- Single-letter upper bound (El Gamal and Cover 82)

$$R(d_1, d_2) \leq I(S; U, V) + I(U; V)$$

where $\mathbb{E}[d(S, \phi(U))] \leq d_1$, $\mathbb{E}[d(S, \varphi(V))] \leq d_1$, $\mathbb{E}[d(S, \psi(U, V))] \leq d_2$. This bound is tight in the no-excess sum rate case (Ahlswede 85), but not tight in general (Zhang and Berger 87).

- Multi-letter lower bound

$$\begin{aligned} nR(d_1, d_2) &= \log|\mathcal{C}_1| + \log|\mathcal{C}_2| \\ &\geq H(f_1^{(n)}(S^n)) + H(f_2^{(n)}(S^n)) \\ &= H(f_1^{(n)}(S^n), f_2^{(n)}(S^n)) + I(f_1^{(n)}(S^n); f_2^{(n)}(S^n)) \\ &= I(S^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) + I(f_1^{(n)}(S^n); f_2^{(n)}(S^n)). \end{aligned}$$

where $f_1^{(n)}$ and $f_2^{(n)}$ meet the distortion constraints.

Remote Source

- The remote-source method (Ozarow 80): introduce a remote source Z_1^n

$$\begin{aligned}
&I(f_1^{(n)}(S^n); f_2^{(n)}(S^n)) \\
&= I(Z_1^n; f_1^{(n)}(S^n)) + (Z_1^n; f_2^{(n)}(S^n)) - I(Z_1^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) \\
&\quad + I(f_1^{(n)}(S^n); f_2^{(n)}(S^n)|Z_1^n) \\
&\geq I(Z_1^n; f_1^{(n)}(S^n)) + (Z_1^n; f_2^{(n)}(S^n)) - I(Z_1^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
nR(d_1, d_2) &\geq I(S^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) + I(f_1^{(n)}(S^n); f_2^{(n)}(S^n)) \\
&\geq I(S^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) - I(Z_1^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) \\
&\quad + I(Z_1^n; f_1^{(n)}(S^n)) + (Z_1^n; f_2^{(n)}(S^n)).
\end{aligned}$$

- Single-letterization

$$\begin{aligned} & I(Z_1^n; f_i^{(n)}(S^n)) \\ & \geq I(Z_1^n; \hat{S}_{\{i\}}^n) \\ & \geq nI(Z_1(T); \hat{S}_{\{i\}}(T)) \end{aligned}$$

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$$\begin{aligned} nR(d_1, d_2) &\geq I(S^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) + I(f_1^{(n)}(S^n); f_2^{(n)}(S^n)) \\ &\geq I(S^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) - I(Z_1^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) \\ &\quad + I(Z_1^n; f_1^{(n)}(S^n)) + (Z_1^n; f_2^{(n)}(S^n)). \end{aligned}$$

In the quadratic Gaussian case, the red part can be single-letterized using the entropy power inequality (Ozarow 80) or the additive noise lemma (Wang and Viswanath 07).

- Single-letterization

$$
\begin{aligned}
& I(S^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) - I(Z_1^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) \\
&= I(Z_1^n, S^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) - I(Z_1^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n)) \\
&= I(S^n; f_1^{(n)}(S^n), f_2^{(n)}(S^n) | Z_1^n) \\
&\geq I(S^n; \hat{S}_{\{1\}}^n, \hat{S}_{\{2\}}^n, \hat{S}_{\{1,2\}}^n | Z_1^n) \\
&= \sum_{t=1}^{n} I(S(t); \hat{S}_{\{1\}}^n, \hat{S}_{\{2\}}^n, \hat{S}_{\{1,2\}}^n | Z_1^n, S^{t-1}) \\
&= \sum_{t=1}^{n} I(S(t); \hat{S}_{\{1\}}^n, \hat{S}_{\{2\}}^n, \hat{S}_{\{1,2\}}^n, Z_1^n, S^{t-1} | Z_1(t)) \\
&\geq \sum_{t=1}^{n} I(S(t); \hat{S}_{\{1\}}(t), \hat{S}_{\{2\}}(t), \hat{S}_{\{1,2\}}(t) | Z_1(t)) \\
&\geq n I(S(T); \hat{S}_{\{1\}}(T), \hat{S}_{\{2\}}(T), \hat{S}_{\{1,2\}}(T) | Z_1(T)).
\end{aligned}
$$

- Introduce $\underline{Z} = (Z_0, Z_1, \cdots, Z_L)$ such that $Z_0 \leftrightarrow Z_1 \leftrightarrow \cdots \leftrightarrow Z_L \leftrightarrow S$ form a Markov chain.

$$R([d]) \geq r([d]) \triangleq \sup_{p_{\underline{Z}|S} \in \mathcal{P}} \inf_{p_{[\hat{S}]|S} \in \mathcal{P}([d])} \Phi(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S})$$

where

$$\Phi(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}) = \sum_{k=1}^{L} \frac{L}{k\binom{L}{k}} \sum_{\mathcal{A} \in 2_+^{\mathcal{L}}, |\mathcal{A}|=k} I(Z_k; \hat{S}_{\mathcal{B}}, \mathcal{B} \in 2_+^{\mathcal{A}} | Z_{k-1}),$$

$$\mathcal{P} = \{p_{\underline{Z}|S} : Z_0 \leftrightarrow Z_1 \leftrightarrow \cdots \leftrightarrow Z_L \leftrightarrow S\},$$

$$\mathcal{P}([d]) = \{p_{\underline{\hat{S}}|S} : \mathbb{E}[d(S, \hat{S}_{\mathcal{A}})] \leq d_{\mathcal{A}}, \mathcal{A} \in 2_+^{\mathcal{L}}\}.$$

A Single-Letter Lower Bound on $R(\underline{d})$

- Introduce $\underline{Z} = (Z_0, Z_1, \cdots, Z_L)$ such that $Z_0 \leftrightarrow Z_1 \leftrightarrow \cdots \leftrightarrow Z_L \leftrightarrow S$ form a Markov chain.

$$R(\underline{d}) \geq r(\underline{d}) \triangleq \max_{p_{\underline{Z}|S} \in \mathcal{P}} \min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \Psi(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}),$$

where

$$\Psi(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}) = \sum_{k=1}^{L} \frac{L}{k} I(Z_k; \hat{S}_k | Z_{k-1}),$$

$$\mathcal{P} = \{p_{\underline{Z}|S} : Z_0 \leftrightarrow Z_1 \leftrightarrow \cdots \leftrightarrow Z_L \leftrightarrow S\},$$

$$\mathcal{P}(\underline{d}) = \{p_{\underline{\hat{S}}|S} : \mathbb{E}[d(S, \hat{S}_k)] \leq d_k, k = 1, \cdots, L\}.$$

- Subset entropy inequality (Han 78)

$$\frac{1}{(k-1)\binom{L}{k-1}} \sum_{\mathcal{A}\in 2^{\mathcal{L}}_{+}, |\mathcal{A}|=k-1} H(f_i^{(n)}(S^n), i\in\mathcal{A}|Z_{k-1}^n)$$

$$\geq \frac{1}{k\binom{L}{k}} \sum_{\mathcal{A}\in 2^{\mathcal{L}}_{+}, |\mathcal{A}|=k} H(f_i^{(n)}(S^n), i\in\mathcal{A}|Z_{k-1}^n)$$

- The Markov chain is required for technical reasons.

- Remove the Markov ordering (replace $\mathcal{P}$ with $\mathcal{Q}$)

$$\max_{p_{\underline{Z}|S} \in \mathcal{Q}} \min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \Psi(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}).$$

- Prove the existence of a saddle-point solution $(p_{\underline{Z}^*|S}, p_{\underline{\hat{S}}^*|S})$ such that

$$\begin{aligned} &\max_{p_{\underline{Z}|S} \in \mathcal{Q}} \min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \Psi(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}) \\ &= \min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \max_{p_{\underline{Z}|S} \in \mathcal{Q}} \Psi(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}) \\ &= \Psi(p_{\underline{Z}^*|S}, p_{\underline{\hat{S}}^*|S}). \end{aligned}$$

- Hope that the Markov ordering is automatically satisfied by the saddle-point solution.

- Evaluate $r(\underline{d})$ for
  - Binary Source: $\mathcal{S} = \{0, 1\}$ and $p_S(0) = p_S(1) = \frac{1}{2}$
  - Erasure Distortion Measure: $\hat{S} = \{0, 1, e\}$ and

$$m_E(s, \hat{s}) = \begin{cases} 0, & s = \hat{s} \\ 1, & \hat{s} = e \\ \infty, & (s, \hat{s}) = (0, 1) \text{ or } (s, \hat{s}) = (1, 0) \end{cases};$$

- Theorem

$$r(\underline{d}) = \max_{\underline{q} \in [0, \frac{1}{2}]^{L+1}} \min_{\underline{\delta} \in \mathcal{D}(\underline{d})} \sum_{k=1}^{L} \frac{L}{k}(1 - \delta_k)[H_b(q_{k-1}) - H_b(q_k)]$$

$$= \min_{\underline{\delta} \in \mathcal{D}(\underline{d})} \max_{\underline{q} \in [0, \frac{1}{2}]^{L+1}} \sum_{k=1}^{L} \frac{L}{k}(1 - \delta_k)[H_b(q_{k-1}) - H_b(q_k)],$$

where $H_b(q) = -q \log q - (1 - q) \log(1 - q)$ and $\mathcal{D}(\underline{d}) = [0, d1] \times \cdots \times [0, d_L]$.

- Remove Markov condition: Let $\mathcal{Q}$ denote the set of all possible conditional distributions $p_{\underline{Z}|S}$, where $\underline{Z} = (Z_0, Z_1, \cdots, Z_L)$. Define

$$\gamma(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}) = \sum_{k=1}^{L} \frac{L}{k}[I(Z_k; \hat{S}_k) - I(Z_{k-1}; \hat{S}_k)]$$

and it is assumed that $\underline{Z} \leftrightarrow S \leftrightarrow \underline{\hat{S}}$ form a Markov chain.

- Notice that

$$r(\underline{d}) = \max_{p_{\underline{Z}|S} \in \mathcal{P}} \min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \gamma(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S})$$

namely

$$\max_{p_{\underline{Z}|S} \in \mathcal{P}} \min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \gamma(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}) = \max_{p_{\underline{Z}|S} \in \mathcal{P}} \min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \sum_{k=1}^{L} \frac{L}{k} I(Z_k; \hat{S}_k | Z_{k-1})$$

Binary Unifrom Source with Erasure Distortion Measure

- Step 1:

$$\max_{p_{\underline{Z}|S}\in\mathcal{Q}} \min_{p_{\underline{\hat{S}}|S}\in\mathcal{P}(\underline{d})} \gamma(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S})$$

$$= \max_{\underline{q}\in[0,\frac{1}{2}]^{L+1}} \min_{\underline{\delta}\in\mathcal{D}(\underline{d})} \sum_{k=1}^{L} \frac{L}{k}(1-\delta_k)[H_b(q_{k-1}) - H_b(q_k)]$$

$$= \min_{\underline{\delta}\in\mathcal{D}(\underline{d})} \max_{\underline{q}\in[0,\frac{1}{2}]^{L+1}} \sum_{k=1}^{L} \frac{L}{k}(1-\delta_k)[H_b(q_{k-1}) - H_b(q_k)],$$

- Step 2:

$$\max_{p_{\underline{Z}|S}\in\mathcal{P}} \min_{p_{\underline{\hat{S}}|S}\in\mathcal{P}(\underline{d})} \gamma(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S}) = \max_{p_{\underline{Z}|S}\in\mathcal{Q}} \min_{p_{\underline{\hat{S}}|S}\in\mathcal{P}(\underline{d})} \gamma(p_{\underline{Z}|S}, p_{\underline{\hat{S}}|S})$$

Step 1:

- Max problem: $\max_{p_{\underline{Z}|S}\in\mathcal{Q}} \gamma(p_{\underline{Z}|S}, p_{\underline{\hat{S}}^*|S})$ is equivalent to

$$\max_{p_{Z_k|S}} -\frac{L}{k}H(\hat{S}_k^*|Z_k) + \frac{L}{k+1}H(\hat{S}_{k+1}^*|Z_k), \quad k = 0, \cdots, L.$$

When $p_{\underline{\hat{S}}^*|S}$ is given as BECs, the maximum value attained by BSC $p_{Z_k|S}$, $k = 0, \cdots, L$ and

$$\max_{p_{\underline{Z}|S}\in\mathcal{Q}} \gamma(p_{\underline{Z}|S}, p_{\underline{\hat{S}}^*|S}) = \max_{\underline{q}\in[0,\frac{1}{2}]^{L+1}} \sum_{k=1}^{L} \frac{L}{k}(1-\delta_k)[H_b(q_{k-1}) - H_b(q_k)]$$

where the optimal solution is attained by $p_{\underline{Z}|S}$ that $p_{Z_{k-1}|S}$ is stochastically degraded with respect to $p_{Z_k|S}$.

The Max problems

$$\max_{q_0\in[0,\frac{1}{2}]} \alpha_1(1-\delta_1)H_b(q_0),$$

$$\max_{q_k\in[0,\frac{1}{2}]} [-\alpha_k(1-\delta_k)+\alpha_{k+1}(1-\delta_{k+1})]H_b(q_k), \quad k=1,\cdots,L-1,$$

$$\max_{q_L\in[0,\frac{1}{2}]} -\alpha_L(1-\delta_L)H_b(q_L)$$

are optimized by the following the maximizers, respectively.

$$q_0 = \begin{cases} 0, & \alpha_1(1-\delta_1)<0 \\ \text{any number in } [0,\frac{1}{2}], & \alpha_1(1-\delta_1)=0 \\ \frac{1}{2}, & \alpha_1(1-\delta_1)>0 \end{cases},$$

$$q_k = \begin{cases} 0, & \alpha_k(1-\delta_k)>\alpha_{k+1}(1-\delta_{k+1}) \\ \text{any number in } [0,\frac{1}{2}], & \alpha_k(1-\delta_k)=\alpha_{k+1}(1-\delta_{k+1}) \\ \frac{1}{2}, & \alpha_k(1-\delta_k)<\alpha_{k+1}(1-\delta_{k+1}) \end{cases}, k=1,...,L-1,$$

$$q_L = \begin{cases} 0, & \alpha_L(1-\delta_L)>0 \\ \text{any number in } [0,\frac{1}{2}], & \alpha_L(1-\delta_L)=0 \\ \frac{1}{2}, & \alpha_L(1-\delta_L)<0 \end{cases}.$$

- Min problem: $\min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \gamma(p_{\underline{Z}^*|S}, p_{\underline{\hat{S}}|S})$ is equivalent to

$$\min_{p_{\hat{S}_k|S}: \mathbb{E}[m_E(\hat{S}_k, S)] \leq d_k} -\frac{L}{k} H(Z_k^*|\hat{S}_k) + \frac{L}{k} H(Z_{k-1}^*|\hat{S}_k), \quad k = 1, \cdots, L.$$

When $p_{\underline{\hat{Z}}^*|S}$ is given as BSCs, the maximum value attained by BEC $p_{\hat{S}_k|S}$, $k = 0, \cdots, L$ and

$$\min_{p_{\underline{\hat{S}}|S} \in \mathcal{P}(\underline{d})} \gamma(p_{\underline{Z}^*|S}, p_{\underline{\hat{S}}|S}) = \min_{\underline{\delta} \in \mathcal{D}(\underline{d})} \sum_{k=1}^{L} \frac{L}{k} (1 - \delta_k)[H_b(q_{k-1}) - H_b(q_k)]$$

Step 1:

The Min problems

$$\min_{\delta_k \in [0, d_k]} \alpha_k (1-\delta_k)[H_b(q_{k-1}) - H_b(q_k)], \quad k = 1, \cdots, L,$$

have the following minimizers.

$$\delta_k = \begin{cases} 0, & \alpha_k[H_b(q_{k-1}) - H_b(q_k)] < 0 \\ \text{any number in } [0, d_k], & \alpha_k[H_b(q_{k-1}) - H_b(q_k)] = 0 \\ d_k, & \alpha_k[H_b(q_{k-1}) - H_b(q_k)] > 0 \end{cases}, \quad k = 1, \cdots, L.$$

Step 1:

- Prove:

$$\max_{\underline{q}\in[0,\frac{1}{2}]^{L+1}} \min_{\underline{\delta}\in\mathcal{D}(\underline{d})} \sum_{k=1}^{L} \frac{L}{k}(1-\delta_k)[H_b(q_{k-1})-H_b(q_k)]$$
$$=\min_{\underline{\delta}\in\mathcal{D}(\underline{d})} \max_{\underline{q}\in[0,\frac{1}{2}]^{L+1}} \sum_{k=1}^{L} \frac{L}{k}(1-\delta_k)[H_b(q_{k-1})-H_b(q_k)]$$

  There exists a saddle-point solution $(\underline{q}^*, \underline{\delta}^*)$ for the above minimax problem.

- Theorem (von Neuman 37): Let $\mathcal{X}$ and $\mathcal{Y}$ be two bounded closed convex sets in the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and $\mathcal{X}\times\mathcal{Y}$ be their Cartesian product in $\mathbb{R}^{m+n}$. Let $\mathcal{U}$ and $\mathcal{V}$ be two closed subsets of $\mathcal{X}\times\mathcal{Y}$ such that for any $x\in\mathcal{X}$ the set $\{y\in\mathcal{Y}:(x,y)\in\mathcal{U}\}$ is non-empty, closed, and convex, and such that for any $y\in\mathcal{Y}$ the set $\{x\in\mathcal{X}:(x,y)\in\mathcal{V}\}$ is non-empty, closed, and convex. Under these assumptions, $\mathcal{U}$ and $\mathcal{V}$ have a common point.

Step 1:

- It can be proved that

$$\max_{p_{\underline{Z}|S} \in \mathcal{Q}} \min_{p_{\hat{\underline{S}}|S} \in \mathcal{P}(\underline{d})} \gamma(p_{\underline{Z}|S}, p_{\hat{\underline{S}}|S}) = \min_{p_{\hat{\underline{S}}|S} \in \mathcal{P}(\underline{d})} \max_{p_{\underline{Z}|S} \in \mathcal{Q}} \gamma(p_{\underline{Z}|S}, p_{\hat{\underline{S}}|S})$$

The optimal solution is attain by $(p_{\underline{Z}^*|S}, p_{\hat{\underline{S}}^*|S})$ which are BSC and BEC respectively, and $p_{Z_{k-1}^*|S}$ is stochastically degraded with respect to $p_{Z_k^*|S}$ (i.e. Markov chain structure $Z_0 \leftrightarrow Z_1 \leftrightarrow \cdots \leftrightarrow Z_L \leftrightarrow S$)

- Recall that

$$\max_{p_{\underline{Z}|S}\in\mathcal{Q}}\min_{p_{\underline{\hat{S}}|S}\in\mathcal{P}(\underline{d})}\gamma(p_{\underline{Z}|S},p_{\underline{\hat{S}}|S})=\min_{p_{\underline{\hat{S}}|S}\in\mathcal{P}(\underline{d})}\max_{p_{\underline{Z}|S}\in\mathcal{Q}}\gamma(p_{\underline{Z}|S},p_{\underline{\hat{S}}|S})$$

is attain by $(p_{\underline{Z}^*|S},p_{\underline{\hat{S}}^*|S})$ that $\underline{Z}^*$ has Markov chain structure, so that

$$\max_{p_{\underline{Z}|S}\in\mathcal{P}}\min_{p_{\underline{\hat{S}}|S}\in\mathcal{P}(\underline{d})}\gamma(p_{\underline{Z}|S},p_{\underline{\hat{S}}|S})=\max_{p_{\underline{Z}|S}\in\mathcal{Q}}\min_{p_{\underline{\hat{S}}|S}\in\mathcal{P}(\underline{d})}\gamma(p_{\underline{Z}|S},p_{\underline{\hat{S}}|S})$$

Hence, we have

$$r(\underline{d})=\max_{p_{\underline{Z}|S}\in\mathcal{Q}}\min_{p_{\underline{\hat{S}}|S}\in\mathcal{P}(\underline{d})}\gamma(p_{\underline{Z}|S},p_{\underline{\hat{S}}|S})$$

- Hamming distortion measure:

$$m_H(s, \hat{s}) = s \oplus_2 \hat{s}$$

- Binary Source with Hamming Distortion Measure:

$$\begin{aligned} r(\underline{d}) &= \max_{\underline{q} \in [0, \frac{1}{2}]^{L+1}} \min_{\underline{\delta} \in \mathcal{D}(\underline{d})} \sum_{k=1}^{L} \frac{L}{k} [H_b(q_{k-1} \odot \delta_k) - H_b(q_k \odot \delta_k)] \\ &= \min_{\underline{\delta} \in \mathcal{D}(\underline{d})} \max_{\underline{q} \in [0, \frac{1}{2}]^{L+1}} \sum_{k=1}^{L} \frac{L}{k} [H_b(q_{k-1} \odot \delta_k) - H_b(q_k \odot \delta_k)] \end{aligned}$$

where $q \odot \delta = q(1 - \delta) + (1 - q)\delta$.

- The Gaussian source (with variance λ) and the mean squared error distortion measure

$$r(\underline{d}) = \max_{\underline{\theta} \in [0,\lambda]^{L+1}} \min_{\underline{\delta} \in \mathcal{D}(d)} \omega(\underline{\theta}, \underline{\delta}) = \min_{\underline{\delta} \in \mathcal{D}(d)} \max_{\underline{\theta} \in [0,\lambda]^{L+1}} \omega(\underline{\theta}, \underline{\delta}),$$

where

$$\omega(\underline{\theta}, \underline{\delta}) = \sum_{k=1}^{L} \frac{L}{2k} \log \left(\frac{\lambda\theta_{k-1} + \lambda\delta_k - \theta_{k-1}\delta_k}{\lambda\theta_k + \lambda\delta_k - \theta_k\delta_k} \right).$$

- Achievable:
  Source-channel erasure codes (Pradhan, Puri, and Ramchandran 04)

- Converse:
  Evaluate the single letter lower bound for the quadratic Gaussian case.

Case 1

- Achievable:
  If $d_k = \lambda$ for all $k < \ell$ and $d_k \geq \left(\frac{k}{\ell} d_\ell^{-1} - \frac{k-\ell}{\ell}\lambda^{-1}\right)^{-1}$ for all $k > \ell$ (with $\ell < L$), then

$$R(\underline{d}) \leq \frac{L}{2\ell} \log\left(\frac{\lambda}{d_\ell}\right).$$

- Converse:
  If $R(\underline{d}) = \frac{L}{2\ell} \log\left(\frac{\lambda}{d_\ell}\right)$ for some $\ell < L$, then

$$d_k \geq \left(\frac{k}{\ell} d_\ell^{-1} - \frac{k-\ell}{\ell}\lambda^{-1}\right)^{-1}, \quad k > \ell.$$

Case 2

- Achievable:
  If $d_k \geq \frac{k}{L} d_L + \frac{L-k}{L}\lambda$ for all $k < L$, then

$$R(\underline{d}) \leq \frac{1}{2}\log\left(\frac{\lambda}{d_L}\right).$$

- Converse:
  If $R(\underline{d}) = \frac{1}{2}\log\left(\frac{\lambda}{d_L}\right)$, then

$$d_k \geq \frac{k}{L} d_L + \frac{L-k}{L}\lambda, \quad k < L.$$

Case 3

- Achievable:
  If $\frac{L}{\ell}d_\ell - \frac{L-\ell}{\ell}\lambda < d_L < \left(\frac{L}{\ell}d_\ell^{-1} - \frac{L-\ell}{\ell}\lambda^{-1}\right)^{-1}$ for some $\ell < L$ and

$$d_k = \lambda, \quad k < \ell,$$
$$d_k \geq \frac{L(k-\ell)(\lambda - d_\ell)d_L + \ell(L-k)(\lambda - d_L)d_\ell}{k(L-\ell)\lambda - L(k-\ell)d_\ell - \ell(L-k)d_L}, \quad \ell < k < L,$$

  then

$$R(\underline{d}) \leq \frac{L}{2\ell}\log\left[\frac{(L-\ell)(\lambda - d_L)}{L(d_\ell - d_L)}\right] + \frac{1}{2}\log\left[\frac{\ell\lambda(d_\ell - d_L)}{(L-\ell)(\lambda - d_\ell)d_L}\right].$$

- Converse:
  If $R(\underline{d}) = \frac{L}{2\ell}\log\left[\frac{(L-\ell)(\lambda-d_L)}{L(d_\ell - d_L)}\right] + \frac{1}{2}\log\left[\frac{\ell\lambda(d_\ell - d_L)}{(L-\ell)(\lambda-d_\ell)d_L}\right]$ for some $\ell < L$ and $\frac{L}{\ell}d_\ell - \frac{L-\ell}{\ell}\lambda < d_L < \left(\frac{L}{\ell}d_\ell^{-1} - \frac{L-\ell}{\ell}\lambda^{-1}\right)^{-1}$, then

$$d_k \geq \frac{L(k-\ell)(\lambda - d_\ell)d_L + \ell(L-k)(\lambda - d_L)d_\ell}{k(L-\ell)\lambda - L(k-\ell)d_\ell - \ell(L-k)d_L}, \quad \ell < k < L.$$

Conclusion

- Single-letterization: the role of auxiliary random variables
- Extremal inequalities
- Minimax theorems
- Information-theoretic limits of Gaussian multiple description coding with two-level distortion constraints

Thank you!