Combinatorial Power Allocation in AC Systems

Approximation, Hardness and Truthfulness for Complex-demand Knapsack Problem

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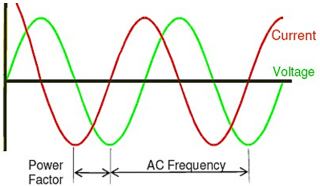
Paper: http://www.SustainableNetworks.org/papers/cks.pdf
Slides: http://www.SustainableNetworks.org/slides/cks.pdf

Story begins with Resource Allocation ...

- Resources are in different forms
  - E.g. time, space, bandwidth, ...
  - and *energy* (electricity is the most common form of energy)
- Smart grid (what is it?)
  - No precise definition, but broadly, modernizing electrical grid using information and communications technology
  - For example, enabling more efficient allocation of energy
- From *communication networking* to *electricity networking*
  - Similarities: Networked structures, Limited storage, Uncertainties in demands and supplies, ...
  - Differences: Homogeneous commodity (i.e. electricity), Periodic quantities (i.e. alternating current/AC)

- Circular motion of dynamo generator $\Rightarrow$ Periodic current and voltage
- Phase between current and voltage



- Complex number representations: $V = |V|e^{\mathbf{i}\omega t}$, $I = |I|e^{\mathbf{i}(\omega t+\theta)}$,
- Power: $P = V \times I$ (also a complex number)
  - Active power: $\mathrm{Re}(P)$
  - Reactive power: $\mathrm{Im}(P)$
  - Apparent power: $|P|$

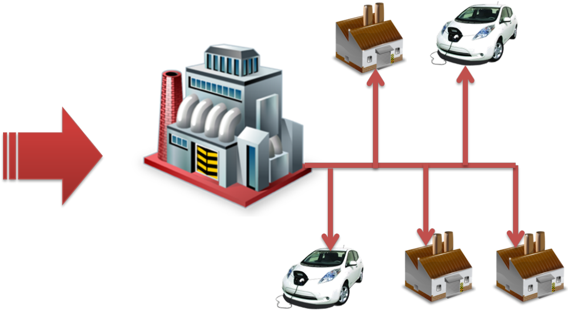
AC Electrical Systems 101 (Lingo)

- Active power ($\mathrm{Re}(P)$)
  - Can do useful work at loads
- Reactive power ($\mathrm{Im}(P)$)
  - Needed to support the transfer of real power over the network
  - Capacitors generate reactive power; inductors to consume it
- Power factor ($\frac{\mathrm{Re}(P)}{|P|}$)
  - Ratio between real power and apparent power
  - Regulations require maximum power factor
- Apparent power ($|P|$)
  - Magnitude of total active and reactive power
  - Cared by power engineers
  - Conductors, transformers and generators must be sized to carry the total current (manifested by apparent power)

Central Problem: Power Allocation

- Utility-maximizing allocation power to end-users
  - Subject to capacity constraints of total apparent power (or current, voltage)
- Elastic (splittable) demands $\Rightarrow$ (Non-)Convex optimization
- Inelastic (unsplittable) demands $\Rightarrow$ Combinatorial optimization
  - Minimum active/reactive power requirement
  - Challenge: Positive reactive power can cancel negative reactive power

From Knapsack to Inelastic Power Allocation



(Traditional) 1D Knapsack Problem

Definition (1DKS)

$$\max \sum_{k \in [n]} x_k u_k$$

subject to

$$\sum_{k \in K} x_k d_k \leq C, \quad x_k \in \{0, 1\} \text{ for } k \in [n]$$

- $[n] := \{1, \ldots, n\}$: a set of users
- u_k: utility of k-th user if its demand is satisfied
- d_k: positive real-valued demand of k-th user
- C: real-valued capacity on total satisfiable demand
- x_k: decision variable of allocation
  - $x_k = 1$, if k-th user's demand is satisfied
  - $x_k = 0$, otherwise

Knapsack Problem for Power Allocation

- Complex-valued resources (e.g. AC power, current, voltage)
  - Discrete optimization mostly concerns real-valued resources
- Allocating complex-valued (AC) power among a set of users
- Inelastic user demands (i.e. fully satisfied or not)
- Maximizing total utility of satisfied users
- Subject capacity constraints
  - Active power and reactive power constraints
  - Apparent power constraint
- Optional:
  - Utility is private information reported by users
  - Selfish users tend to exaggerate their utility

2D Knapsack Problem

Definition (2DKS)

$$\max_{x_k \in \{0,1\}} \sum_{k \in K} x_k u_k \quad (1)$$

subject to

$$\sum_{k \in K} x_k d_k^{\mathrm{R}} \le C^{\mathrm{R}} \text{ and } \sum_{k \in K} x_k d_k^{\mathrm{I}} \le C^{\mathrm{I}} \quad (2)$$

- $d_k^{\mathrm{R}} + \mathbf{i} d_k^{\mathrm{I}}$: complex-valued demand of k-th user
- $C^{\mathrm{R}} + \mathbf{i} C^{\mathrm{I}}$: complex-valued power capacity
  - Real-part: Active power $(d_k^{\mathrm{R}}, C^{\mathrm{R}})$
  - Imaginary-part: Reactive power $(d_k^{\mathrm{I}}, C^{\mathrm{I}})$
- Well-known problem

Complex-demand Knapsack Problem

Definition (CKS)

$$\max \sum_{k \in K} x_k u_k$$

subject to

$$\Big| \sum_{k \in K} x_k d_k \Big| \leq C, \quad x_k \in \{0, 1\} \text{ for } k \in [n]$$

- d_k: complex-valued demand of k-th user ($d_k = d_k^{\mathrm{R}} + \mathbf{i} d_k^{\mathrm{I}}$)
- C: real-valued capacity of total satisfiable demand in apparent power

Complex-demand Knapsack Problem

Definition (CKS)

$$\max \sum_{k \in K} x_k u_k$$

subject to

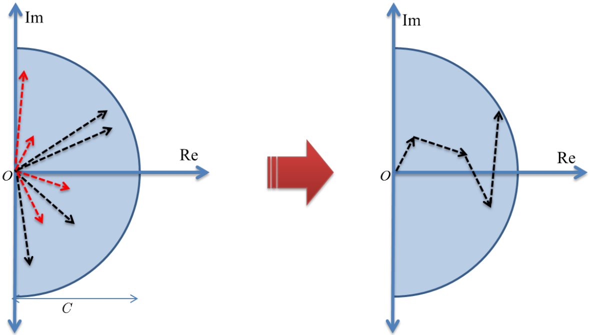
$$\Big| \sum_{k \in K} x_k d_k \Big| \leq C, \quad x_k \in \{0,1\} \text{ for } k \in [n]$$

- It is a 0/1-quadratic programming problem:

$$\begin{aligned} \max \quad & \textstyle\sum_{k \in [n]} x_k u_k \\ \text{s.t.} \quad & (\textstyle\sum_{k \in [n]} d_k^R x_k)^2 + (\textstyle\sum_{k \in [n]} d_k^I x_k)^2 \leq C^2 \\ & x_k \in [0,1] \text{ for all } k \in [n]. \end{aligned}$$

- A new variant of knapsack problem

Complex-demand Knapsack Problem



Pictorially,

- Picking a maximum-utility subset of vectors, such that the sum lies within a circle

Definitions of Approximation Algorithms

- For set S of users, denote by $u(S) \triangleq \sum_{k \in S} u_k$
- Denote S^* an optimal solution of CKS

Definition

For $\alpha \in (0, 1]$ and $\beta \geq 1$, a bi-criteria (α, β)-approximation to CKS is a set S satisfying

$$u(S) \geq \alpha \cdot u(S^*)$$
$$\Big|\sum_{k \in S} d_k\Big| \leq \beta \cdot C$$

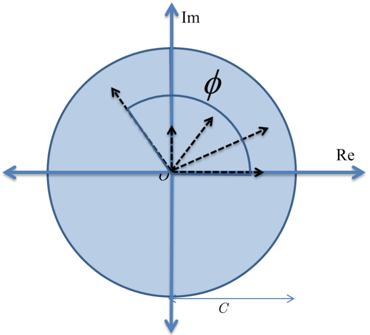
- *Polynomial-time approximation scheme (PTAS)*: an algorithm computes $(1-\epsilon, 1)$-approximation in time polynomial in n for a fixed ϵ
- *Bi-criteria polynomial-time approximation scheme (PTAS)*: an algorithm computes $(1-\epsilon, 1+\epsilon)$-approximation
- *Fully polynomial-time approximation scheme (FPTAS)*: PTAS and additionally requires polynomial running time in $1/\epsilon$

Prior Results for Knapsack Problems

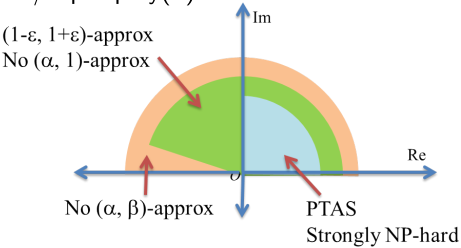
- FPTAS for $1\mathrm{DKS}$
  - Using dynamic programming and scaling (Lawler, 1979)
- No FPTAS for $m\mathrm{DKS}$ where $m \geq 2$
  - Reducing to equipartition problem (Gens and Levner, 1979)
- PTAS for $m\mathrm{DKS}$ where $m \geq 2$
  - Using partial exhaust search and LP (Freize and Clarke, 1985)
- Truthful (monotone) FPTAS for $1\mathrm{DKS}$
  - Monotonicity (Briest, Krysta and Vocking, 2005)
- Truthful bi-criteria FPTAS for multi-minded $m\mathrm{DKS}$
  - Dynamic programming, scaling and VCG (Krysta, Telelis and Ventre, 2013)

Some Definitions

- The problem is invariant under rotation
- Let ϕ be the maximum angle between any two demands
- Denote this restriction by $\mathrm{CKS}[\phi]$
- Write $\mathrm{CKS}[\phi_1, \phi_2]$ for $\mathrm{CKS}[\phi]$ with $\phi \in [\phi_1, \phi_2]$



- Write $\mathrm{CKS}[\phi_1, \phi_2]$ for $\mathrm{CKS}[\phi]$ with $\phi \in [\phi_1, \phi_2]$
- Positive results
  - PTAS for $\mathrm{CKS}[0, \frac{\pi}{2}]$
  - Bi-criteria FPTAS for $\mathrm{CKS}[0, \pi\text{-}\varepsilon]$ for $\varepsilon = 1/\text{poly}(n)$
- Inapproximability results
  - $\mathrm{CKS}[0, \frac{\pi}{2}]$ is strongly NP-hard [Yu and Chau, 2013]
  - Unless P=NP, there is no $(\alpha, 1)$-approximation for $\mathrm{CKS}[\frac{\pi}{2}, \pi]$
  - Unless P=NP, there is no (α, β)-approximation for $\mathrm{CKS}[\pi\text{-}\varepsilon, \pi]$ for some $\varepsilon = 1/\text{super-poly}(n)$

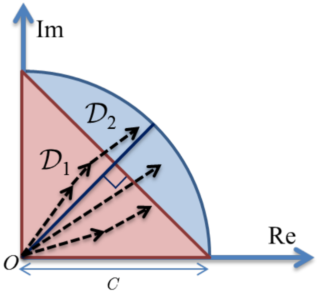


Summary of Results

	$\text{CKS}[0, \frac{\pi}{2}]$	$\text{CKS}[0, \pi\text{-}\varepsilon]$	$\text{CKS}[\pi\text{-}\varepsilon, \pi]$
Pure Inelastic	PTAS No FPTAS	Bi-criteria FPTAS No $(\alpha, 1)$-approx	Bi-criteria Inapproximable
Mixed with Elastic Demands (Linear Utility)	PTAS	Bi-criteria PTAS	
Multi-minded Preferences	PTAS	Bi-criteria FPTAS	
Truthful Mechanism	Randomized PTAS	Deterministic Bi-criteria FPTAS	

Simple Algorithm ($(\frac{1}{2}+\epsilon)$-Approx)

- Assume $\mathrm{CKS}[0, \frac{\pi}{2}]$
- Let S^* be an optimal solution
- Intuition:
  - Case 1: $\sum_{i \in S^*} d_j$ lies in $\mathcal{D}_1$
  - Case 2: $\sum_{i \in S^*} d_j$ lies in $\mathcal{D}_2$ and $|S^*| = 1$
  - Case 3: $\sum_{i \in S^*} d_j$ lies in $\mathcal{D}_2$ and $|S^*| > 1$



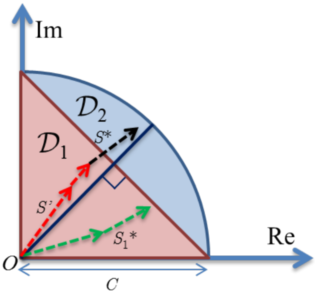
  - Case 1 and Case 2 are easy. And Case 3?

- Case 3: $\sum_{i \in S^*} d_j$ lies in $\mathcal{D}_2$ and $|S^*| > 1$

Lemma

Let S_1^* be an optimal solution within $\mathcal{D}_1$, and S^* be an optimal solution within $\mathcal{D}_1 \cup \mathcal{D}_2$, then

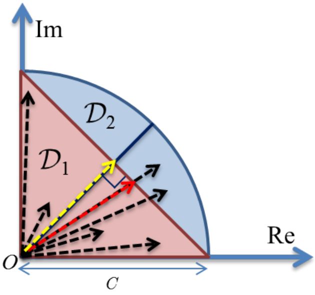
$$\sum_{j \in S^*} u_j \le 2 \sum_{j \in S_1^*} u_j$$



Simple Algorithm ($(\frac{1}{2}+\epsilon)$-Approx)

$(\frac{1}{2}+\epsilon)$-approximation algorithm for CKS$[0, \frac{\pi}{2}]$

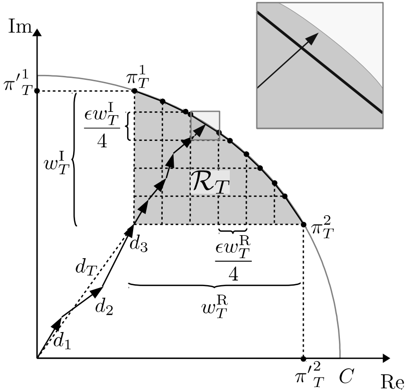
- For each d_j, if d_j lies in $\mathcal{D}_2$, only retain the part in $\mathcal{D}_1$
- Project each d_j onto 1DKS
- Apply FPTAS for 1DKS to solve $\{x_j\}$



PTAS for $\mathrm{CKS}[0, \frac{\pi}{2}]$

- Polygonizing (inscribing polygon within) the circular feasible region
  - Approximate CKS by $m\mathrm{DKS}$
- PTAS for $m\mathrm{DKS}$ with constant m cannot be applied directly
  - Consider optimal solution with large (in magnitude) demands and many small demands, each has the same utility
- Better solution (polygonizing + guessing by partial exhaustive search)
  1. Guess large demands (for a $\frac{1}{\epsilon}$ subset)
  2. Polygonizing by constructing a lattice on the remaining part of the circular region with cell size proportional to ϵ
  3. Find the maximum-utility set of demands in polygonized region (i.e. mDKS problem) where m is a constant depending on $1/\epsilon$
  4. Repeat for every $\frac{1}{\epsilon}$ subset and retain the best solution

PTAS for CKS$[0, \frac{\pi}{2}]$



1. Guess large demands (for a $\frac{1}{\epsilon}$ subset)
2. Polygonizing by constructing a lattice on the remaining part
3. Find the maximum-utility set of demands
4. Repeat for every $\frac{1}{\epsilon}$ subset and retain the best solution

PTAS for CKS$[0, \frac{\pi}{2}]$

CKS-PTAS *for* CKS$[0, \frac{\pi}{2}]$

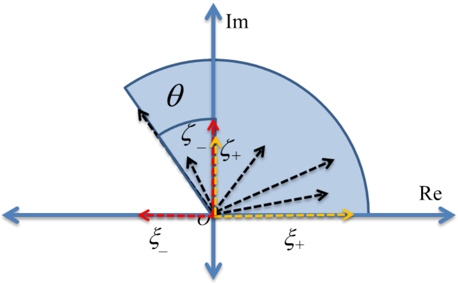
- $\hat{S} \leftarrow \varnothing$
- For each subset $T \subseteq [n]$ of size at most $\min\{n, \frac{1}{\epsilon}\}$
  - Set $d_T \leftarrow \sum_{k \in T} d_k$
  - Obtain $S \leftarrow m\mathrm{DKS\text{-}PTAS}[d_T]$ by polygonization within accuracy ϵ
  - If $u(\hat{S}) < u(S)$,
    - $\hat{S} \leftarrow S$
- Return $\hat{S}$

Theorem

For any $\epsilon > 0$, CKS-PTAS *is a* $(1 - 2\epsilon, 1)$*-approx to* CKS$[0, \frac{\pi}{2}]$

Running time is $n^{O(\frac{1}{\epsilon^2})} \log U$, $U \triangleq \max\left\{C, \max\{d_k^{\mathrm{R}}, d_k^{\mathrm{I}}, u_k \mid k \in [n]\}\right\}$

- $\mathrm{CKS}[0, \frac{\pi}{2}]$ $(\mathrm{Re}(d) \geq 0, \mathrm{Im}(d) \geq 0) \Rightarrow$ no demands cancel others
- $\mathrm{CKS}[0, \pi\text{-}\varepsilon]$ $(\mathrm{Re}(d) \lesseqgtr 0) \Rightarrow$ some demands can cancel others
- But $\theta < \pi$, $\Rightarrow \mathrm{Im}(d) > 0$, when $\mathrm{Re}(d) < 0$
- Intuition:
  - Let $S_+ \triangleq \{k \mid d_k^{\mathrm{R}} \geq 0, k \in S\}$ and $S_- \triangleq \{k \mid d_k^{\mathrm{R}} < 0, k \in S\}$
  - $\xi_+ = \sum_{k \in S_+} d_k^{\mathrm{R}} \leq C(1 + \tan\theta)$, $\quad \zeta_+ = \sum_{k \in S_+} d_k^{\mathrm{I}} \leq C$
  - $\xi_- = \sum_{k \in S_-} -d_k^{\mathrm{R}} \leq C \tan\theta$, $\quad \zeta_- = \sum_{k \in S_-} d_k^{\mathrm{I}} \leq C$



- Basic Ideas:
  1. Enumerate the guessed total projections on real and imaginary axes for S_+ and S_- respectively
  2. Assume that $\tan\theta$ is polynomial in n
  3. Then solve two separate $2\mathrm{DKS}$ exact problems that satisfy $(\xi_+ - \xi_-)^2 + (\zeta_+ + \zeta_-)^2 \le C^2$
     - One in the first quadrant, while another in the second quadrant
  4. But $2\mathrm{DKS}$ exact is generally NP-Hard
     - Similar to bi-criteria FPTAS in $m\mathrm{DKS}$
     - By scaling and truncating the demands makes the approximate problem solvable efficiently by dynamic programming
     - But violation is allowed $\Rightarrow$ bi-criteria FPTAS

Bi-criteria FPTAS for CKS$[0, \pi\text{-}\varepsilon]$

CKS-biFPTAS *for* CKS$[0, \pi\text{-}\varepsilon]$

- For all d_k and $k \in [n]$
  - Set $\hat{d}_k \leftarrow \hat{d}_k^{\mathrm{R}} + \mathbf{i}\hat{d}_k^{\mathrm{I}} \triangleq \left\lceil \frac{d_k^{\mathrm{R}}}{L} \right\rceil + \mathbf{i} \left\lceil \frac{d_k^{\mathrm{I}}}{L} \right\rceil$
- For all $\xi_+ \in \mathcal{A}_+, \xi_- \in \mathcal{A}_-, \zeta_+, \zeta_- \in \mathcal{B}$
  - If $(\xi_+ - \xi_-)^2 + (\zeta_+ + \zeta_-)^2 \le C^2$
    - $F_+ \leftarrow \text{2DKS-Exact}[\{u_k, \hat{d}_k\}, \frac{\xi_+}{L}, \frac{\zeta_+}{L}]$
    - $F_- \leftarrow \text{2DKS-Exact}[\{u_k, \hat{d}_k\}, \frac{\xi_-}{L}, \frac{\zeta_-}{L}]$
    - If $F_+, F_- \neq \varnothing$ and $u(F_+ \cup F_-) > u(\hat{S})$
    - $\hat{S} \leftarrow \{F_+ \cup F_-\}$
- Return $\hat{S}$

Theorem

For any $\epsilon > 0$, CKS-biFPTAS *is* $(1, 1+\epsilon)$*-approximation for* CKS$[0, \pi\text{-}\varepsilon]$. *Running time is polynomial in both* n, $\frac{1}{\epsilon}$ *and* $\tan\theta$.

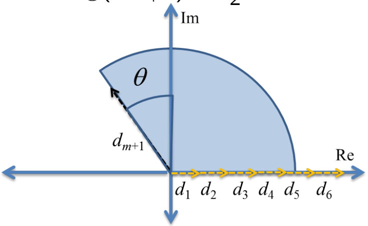
Inapproxability of CKS[π-ε, π]

Theorem

Unless P=NP,

- *No* $(\alpha, 1)$*-approximation for* CKS$[\frac{\pi}{2}+\varepsilon, \pi]$ *where* α, ε *have polynomial length in n*
- *No* (α, β)*-approximation for* CKS$[\pi$-$\varepsilon, \pi]$*, where* α *and* β *have polynomial length, and* ε *depends exponentially on n.*

- Hardness hold even if all demands are on the real line, except one demand d_{m+1} such that $\arg(d_{m+1}) = \frac{\pi}{2} + \theta$, for some $\theta \in [0, \frac{\pi}{2}]$



Proof Ideas:

- Subset sum problem (SubSum):
  - An instance I is a set of positive integers $A \triangleq \{a_1, \ldots, a_m\}$ and positive integer B,
  - Decide if there exist a subset of A that sums-up to exactly B
- Mapping from SubSum to CKS
  - For each $a_k, k = 1, ..., m$, define $d_k \triangleq a_k$
  - Define an additional $d_{m+1} \triangleq -B + \mathbf{i}B \cot \theta$
  - For all $k = 1, ..., m$, let utility $u_k \triangleq \frac{\alpha}{m+1}$, and $u_{m+1} \triangleq 1$
  - Let $C \triangleq B \cot \theta$.
- Showing equivalence
  - SubSum(I) is feasible $\Rightarrow$ There is an (α, β)-approximation solution of utility at least α to CKS
  - There is (α, β)-approximation solution of utility at least α to CKS $\Rightarrow$ There is an feasible solution to SubSum(I)

Proof Ideas:

- Suppose there is (α, β)-approximation solution to CKS
- Since user $m+1$ has utility $u_{m+1} = 1$ and the rest of users utilities $\sum_{k=1}^{m} u_k < \alpha$, user $m+1$ must be included
- Therefore,

$$(\sum_{k=1}^{m} d_k^{\mathrm{R}} x_k - B)^2 + B^2 \cot^2 \theta \leq \beta^2 C^2$$

$$(\sum_{k=1}^{m} d_k^{\mathrm{R}} x_k - B)^2 \leq \beta^2 C^2 - B^2 \cot^2 \theta = B^2 \cot^2 \theta(\beta^2 - 1)$$

- $\textsc{SubSum}$ is feasible, iff $|\sum_{k=1,\ldots,m} a_k x_k - B| < 1$
- $\textsc{SubSum}(I)$ is feasible when $B^2 \cot^2 \theta(\beta^2 - 1) < 1$
  - This occurs when $\beta = 1$, which proves the first claim
  - When θ is large enough such that $B^2 \cot^2 \theta(\beta^2 - 1) < 1$ (i.e., $\theta > \tan^{-1} \sqrt{B^2(\beta^2 - 1)}$, where B is not polynomial in n), which proves the second claim

Furthermore, Extensions of Basic Results

1. Mixing elastic and inelastic demands (some x_k are fractional)
    - Combining demands with splittable and unsplittable demands
2. Multi-minded preferences
    - More choices over multiple unsplittable demands
3. Randomized truthful in expectation mechanisms for $\mathrm{CKS}[0, \frac{\pi}{2}]$
    - Incentivizing users to report true utilities and demands
4. Networked setting of inelastic power allocation
    - Sharing in electrical grid, Constrained by edge capacities

Mixing Elastic and Inelastic Demands

- Let $\mathcal{N}$ be the set of users with inelastic demands
- Let $\mathcal{E}$ be the set of users with elastic demands
  - Linear utility function
  - Utility of satisfying a demand $d_k x_k$ where $x_k \in [0,1]$ is represented by $u_k x_k$, where u_k is maximum utility
- New optimization problem

$$
\begin{aligned}
(\mathrm{CKS}_{\mathrm{mx.lin}}) \qquad & \max \sum_{k \in \mathcal{N} \cup \mathcal{E}} u_k x_k \\
\text{subject to} \qquad & \Big| \textstyle\sum_{k \in \mathcal{N} \cup \mathcal{E}} d_k x_k \Big| \le C \\
& x_k \in \{0,1\} \text{ for all } k \in \mathcal{N} \text{ and} \\
& x_k \in [0,1] \text{ for all } k \in \mathcal{E}.
\end{aligned}
$$

- We extend PTAS and bi-criteria FPTAS of CKS to $\mathrm{CKS}_{\mathrm{mx.lin}}$, by first solving a convex programming problem

Multi-minded Preferences

- Non-single minded preferences: $\mathcal{D}$ is a set of feasible demands
- Each agent can express multiple preferences over more than one unsplittable demand

$$
\begin{aligned}
&(\textsc{nsm}\text{CKS}) \qquad \max \sum_{k\in\mathcal{N}} \sum_{d\in\mathcal{D}} v_k(d) x_{k,d} \\
&\text{subject to} \quad \Big(\sum_{k\in\mathcal{N}} \sum_{d\in\mathcal{D}} d^{\mathrm{R}} \cdot x_{k,d}\Big)^2 + \Big(\sum_{k\in\mathcal{N}} \sum_{d\in\mathcal{D}} d^{\mathrm{I}} \cdot x_{k,d}\Big)^2 \le C^2 \\
&\qquad \textstyle\sum_{d\in\mathcal{D}} x_{k,d} = 1, \qquad \text{for all } k \in \mathcal{N} \\
&\qquad x_{k,d} \in \{0,1\} \text{ for all } k \in \mathcal{N}.
\end{aligned}
$$

- Multi-minded preferences:

$$
v_k(d) = \begin{cases} \max_{d_k\in\mathcal{D}_k}\{v_k(d_k) : \ |d_k^{\mathrm{R}}| \ge |d^{\mathrm{R}}|, |d_k^{\mathrm{I}}| \ge |d^{\mathrm{I}}|, \\ \quad \mathrm{sgn}(d_k^{\mathrm{R}}) = \mathrm{sgn}(d^{\mathrm{R}}), \mathrm{sgn}(d_k^{\mathrm{I}}) = \mathrm{sgn}(d^{\mathrm{I}})\} & \text{if } d_k \in D_k, \\ 0, & \text{otherwise} \end{cases}
$$

- Let $\mathcal{V} \triangleq \mathcal{V}_1 \times \cdots \times \mathcal{V}_n$, where $\mathcal{V}_i$ is the set of all possible valuations of user i, and let Ω be a set of outcomes
- A randomized mechanism $(\mathcal{A}, \mathbb{P})$ is defined by
  - An allocation rule $\mathcal{A} : \mathcal{V} \to \mathcal{D}(\Omega)$
  - A payment rule $\mathbb{P} : \mathcal{V} \to \mathcal{D}(\Re^n_+)$, where $\mathcal{D}(\mathcal{S})$ denotes the set of probability distributions over set $\mathcal{S}$
- The utility of player i when it receives the vector of bids $v \triangleq (v_1, ..., v_n) \in \mathcal{V}$, is the random variable $U_k(v) = \bar{v}_k(x(v)) - p_i(v)$,
  - $x(v) \sim \mathcal{A}(v)$, and $p(v) = (p_1(v), ..., p_n(v)) \sim \mathbb{P}(v)$;
  - $\bar{v}_i$ denotes the true valuation of player i.
- A randomized mechanism is said to be *truthful in expectation*,
  - If for all i and all $\bar{v}_i, v_i \in \mathcal{V}_i$, and $v_{-k} \in \mathcal{V}_{-k}$, it guarantees that $\mathbb{E}[U_k(\bar{v}_k, v_{-k})] \geq \mathbb{E}[U_k(v_k, v_{-k})]$, when the true and reported valuations of player k are $\bar{v}_k$ and v_k, respectively

Truthful Mechanisms

Definition

- Abstractly speaking, the feasible set of a problem is a convex set $\mathcal{X} \subseteq [0,1]^n$ for the relaxed version without integral constraints or $\mathcal{X}^{\mathcal{N}} \triangleq \{x \in \mathcal{X} \mid x_k \in \{0,1\} \text{ for all } k \in \mathcal{N}\}$ with integral constraints
- For a convex polytope $\mathcal{Q} \subseteq [0,1]^n$, we define $\beta \cdot \mathcal{Q} \triangleq \{\beta \cdot x \mid x \in \mathcal{Q}\}$
- An algorithm is called an (α, β)-LP-based approximation for $\mathcal{Q}^{\mathcal{N}}$, if for any $u \in \mathbb{R}^n_+$, it returns in polynomial time an $\hat{x} \in (\beta \cdot \mathcal{Q})^{\mathcal{N}}$, such that $u^T \hat{x} \geq \alpha \cdot \max_{x \in \mathcal{Q}} u^T x$

Theorem (Lavi-Swamy 2005)

If $\mathcal{Q}$ is a convex polytope satisfying the packing property and admitting and α-LP-based approximation algorithm for $\mathcal{Q}^{\mathcal{N}}$. Then one can construct a randomized, individually rational, α-socially efficient mechanism on the set of outcomes $\mathcal{Q}^{\mathcal{N}}$, that is truthful-in-expectation and has no positive transfer.

Truthful Mechanisms

- We extend the Lavi-Swamy theorem to non-linear problem (e.g. complex-demand knapsack problem CKS)
- CKS can be approximated by LP subproblems when $\mathrm{CKS}[0, \frac{\pi}{2}]$
- We show that there is PTAS for $\mathrm{CKS}[0, \frac{\pi}{2}]$ that admits a randomized, individually rational, α-socially efficient mechanism on the set of outcomes $\mathcal{Q}^{\mathcal{N}}$, that is truthful-in-expectation and has no positive transfer
- Our results can be generalized to other non-linear problems
- Furthermore, we use VCG and dynamic programming to construct a truthful PTAS for $\mathrm{CKS}[0, \pi\text{-}\varepsilon]$

Networked Setting of Inelastic Power Allocation

- Networked power flow is a difficult problem (non-convex)
- A simplified model of electrical grid $\mathcal{G} = (N, E)$
- Load $k \in \mathcal{R}$ has an internal impedance Z_{u_k} between its nodal voltage V_{u_k} and the ground, and requires an inelastic power demand d_k
- Consider a single source of generator at node $u_G \in N$
- We assume that the generation power is not limited and hence can feasibly support all loads, if not limited by edge capacity

$$(\textsc{NetP}) \qquad \max_{x_k \in \{0,1\}} \sum_{k \in \mathcal{R}} u_k x_k$$

$$\text{subject to} \qquad \frac{V_{u_k}^2}{Z_{u_k}} = x_k d_k \text{ for all } k \in \mathcal{R}$$

$$V_u - V_v = I_{(u,v)} Z_{(u,v)} \text{ for all } (u, v) \in E$$

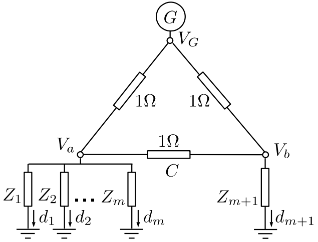
$$\textstyle\sum_{v:\text{Neighbor}(u)} I_{(u,v)} = 0 \text{ for all } u \neq u_G$$

$$|I_{(u,v)}| \leq C_{(u,v)} \text{ for all } (u, v) \in E$$

Theorem

Unless P=NP, there is no (α, β)-approximation for NETP *(even considering a DC system)*

- We consider the following gadget



- By equivalence of SUBSUM to NETP
- Open question: Then what can we do?

Conclusion and Implications

- A first study of combinatorial power allocation for AC systems
- Thorough approximation and hardness results
- Significance: A first step from communication networking to electricity networking
  - Knapsack $\Rightarrow$ Complex-demand Knapsack
  - Commodity flow problem $\Rightarrow$ Optimal power flow problem
  - Network design problem $\Rightarrow$ Optimal islanding problem
- Open questions
  - Networked power allocation (e.g. tree, grid, star)
  - Coping with inapproximability (relaxing satisfiability)
  - Efficient incentive compatible mechanisms
  - Joint scheduling and power allocation

Paper: `http://www.SustainableNetworks.org/papers/cks.pdf`
Slides: `http://www.SustainableNetworks.org/slides/cks.pdf`