Properties of Network Polynomials

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Outline:

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- Network Coding: An example

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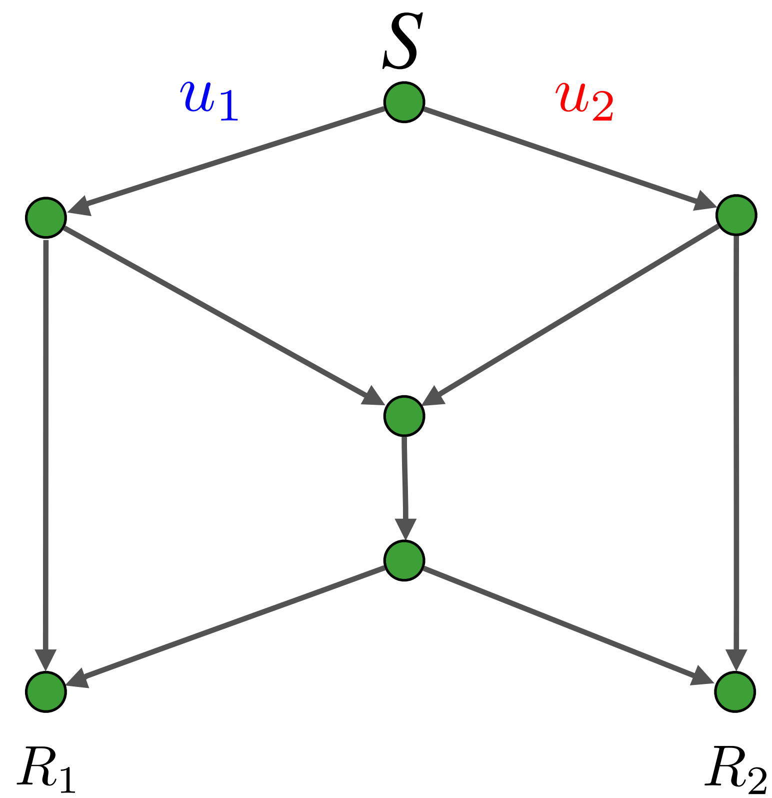
- Network Coding: An example
- Network model and basic definitions
- Problem formulation and motivation
- Network polynomial for 1,2 receivers
- An open problem

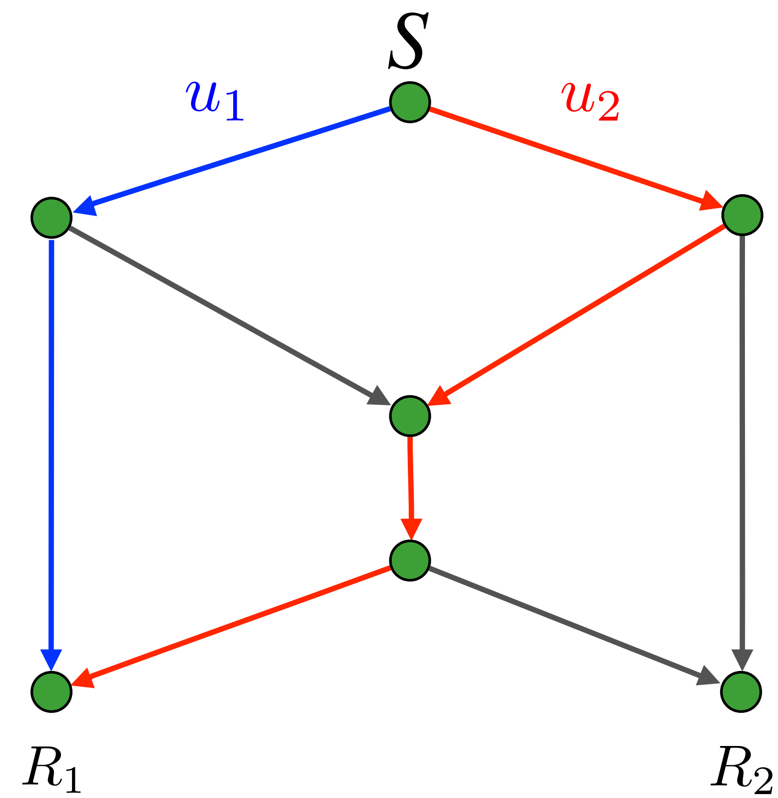
Network Coding:

(Ahlswede, Cai, Li, Yeung 2000) In data transfer over networks, processing the data at the nodes can significantly improve the throughput.

Example: (Butterfly Network, Ahlswede et. al)



Example:



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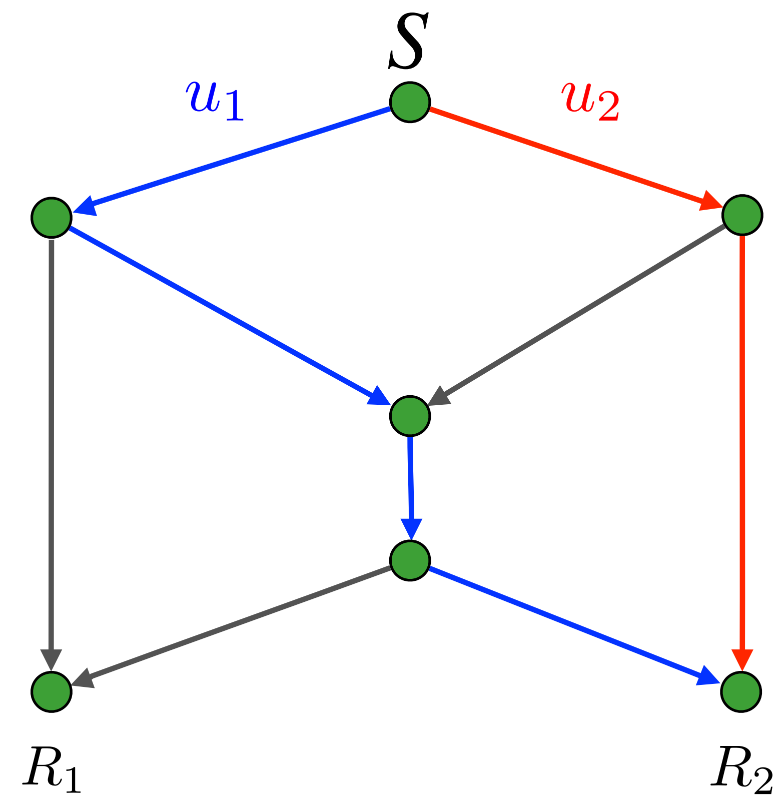

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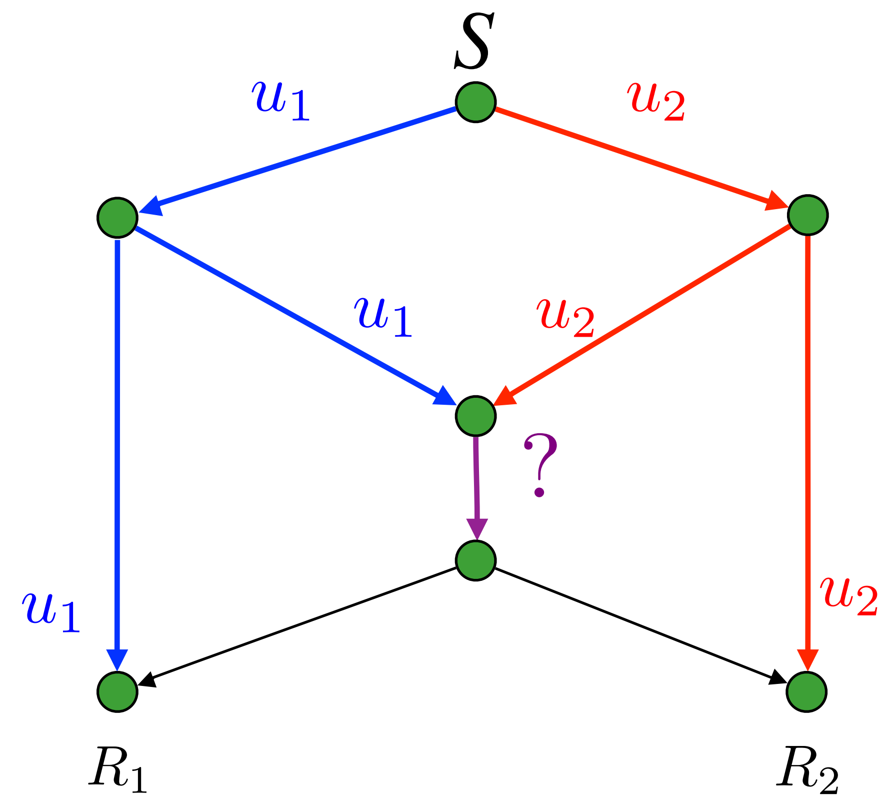

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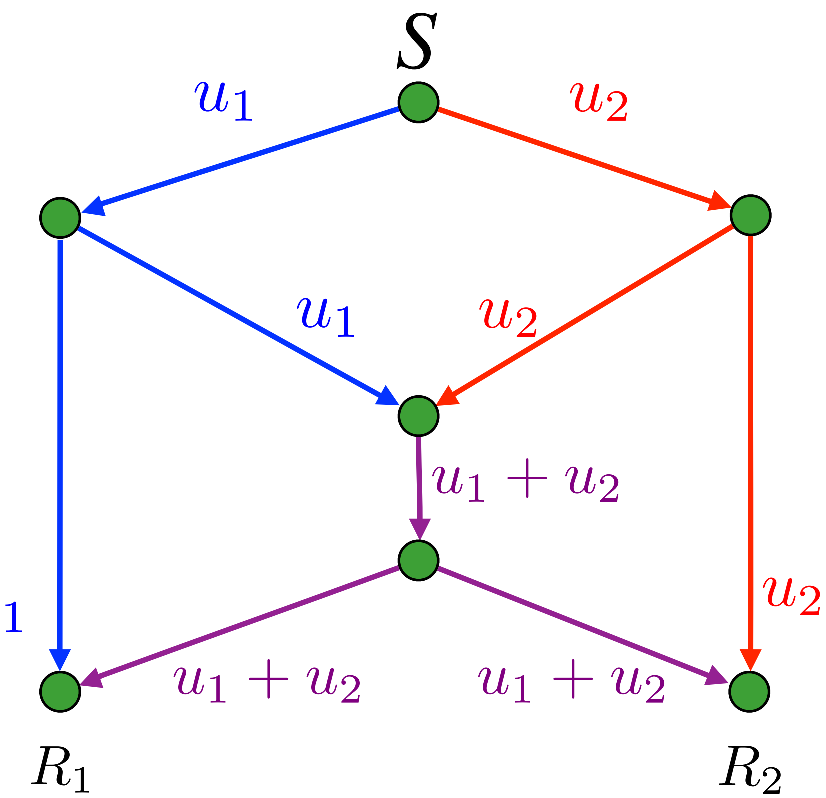
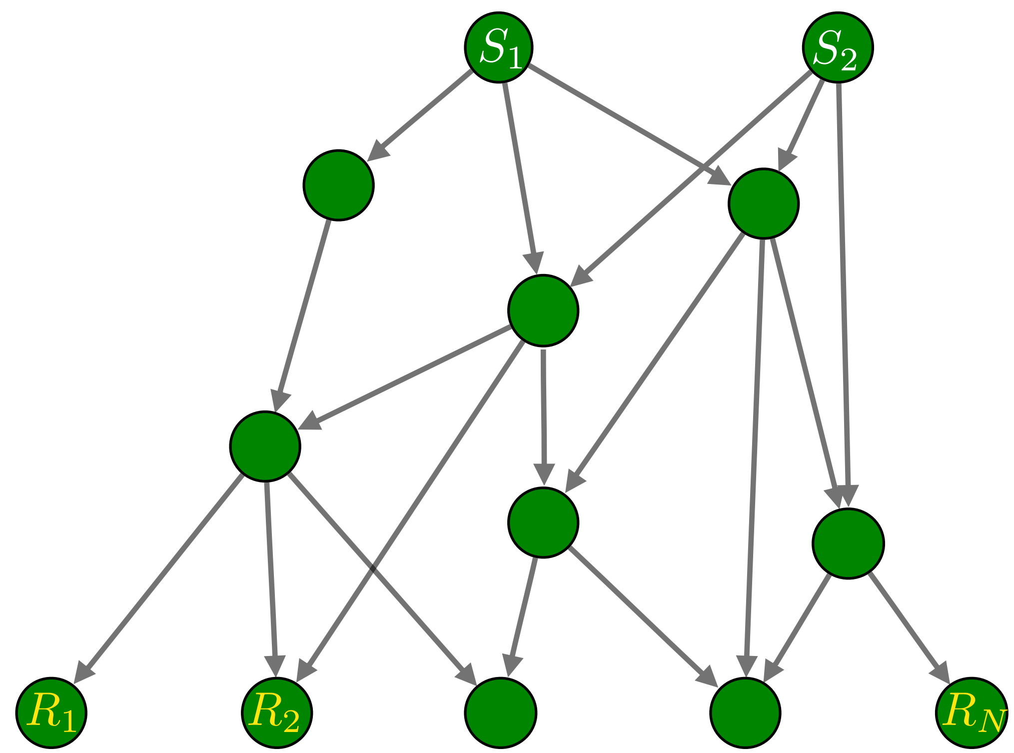
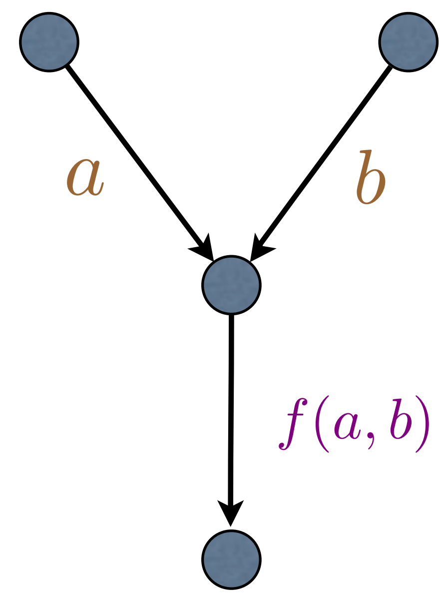


Network Model:



Node Operation:

$a, b \in \chi$

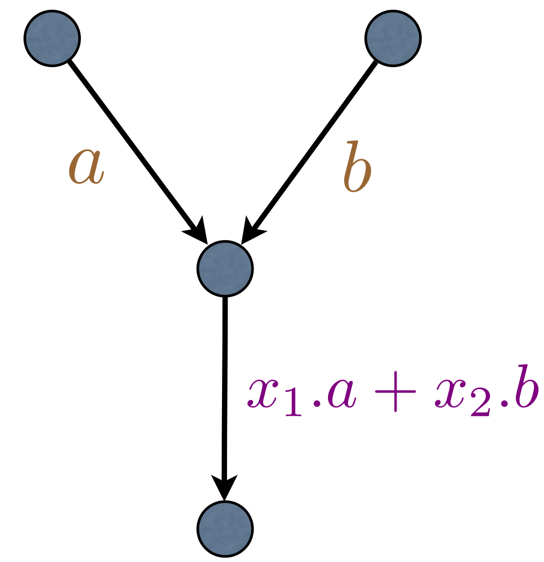


Network Coding Problem:

Designing the node operations such that each receiver can satisfy its demand from the received information.

Linear Node Operation:

$a, b \in \mathbb{F}_q$



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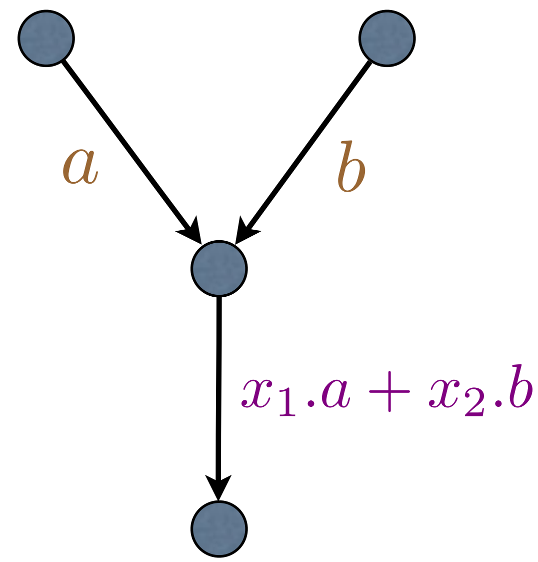

Linear operations ⟺ Linear network codes

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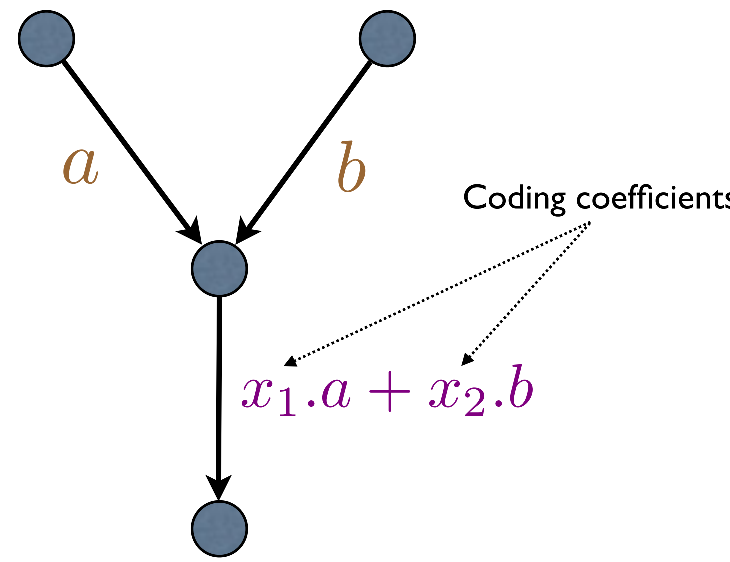


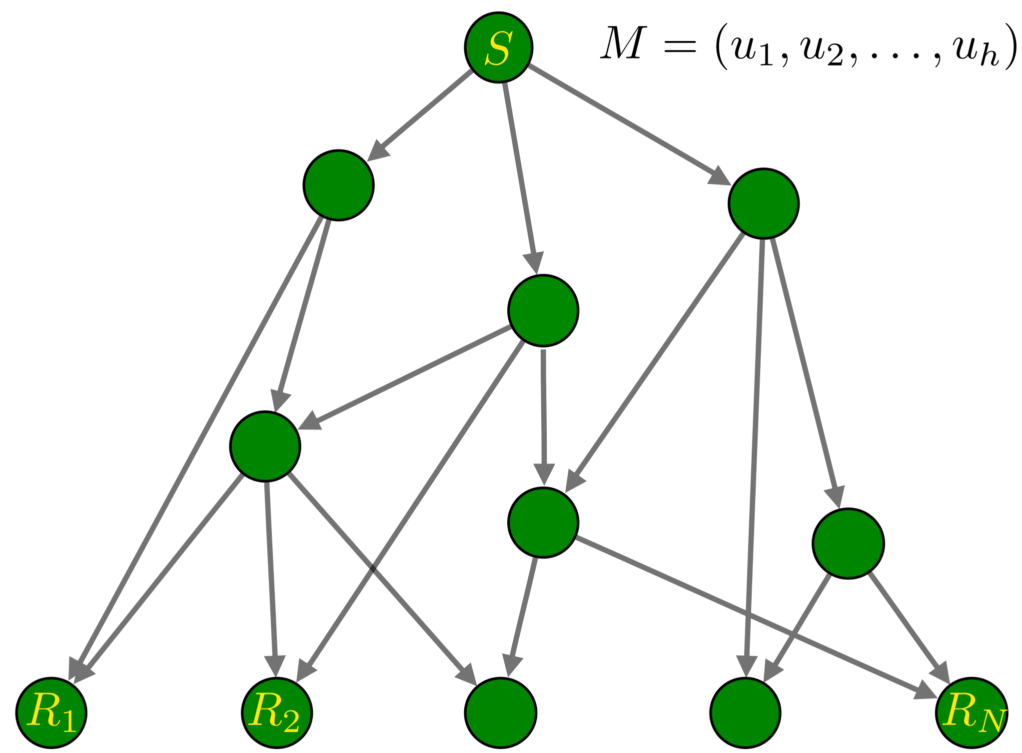
Linear operations ⟺ Linear network codes

Linear Network Coding Problem:

Designing the coding coefficients from appropriate finite field such that each receiver can satisfy its demand from the received messages.

Multicast Network Model:

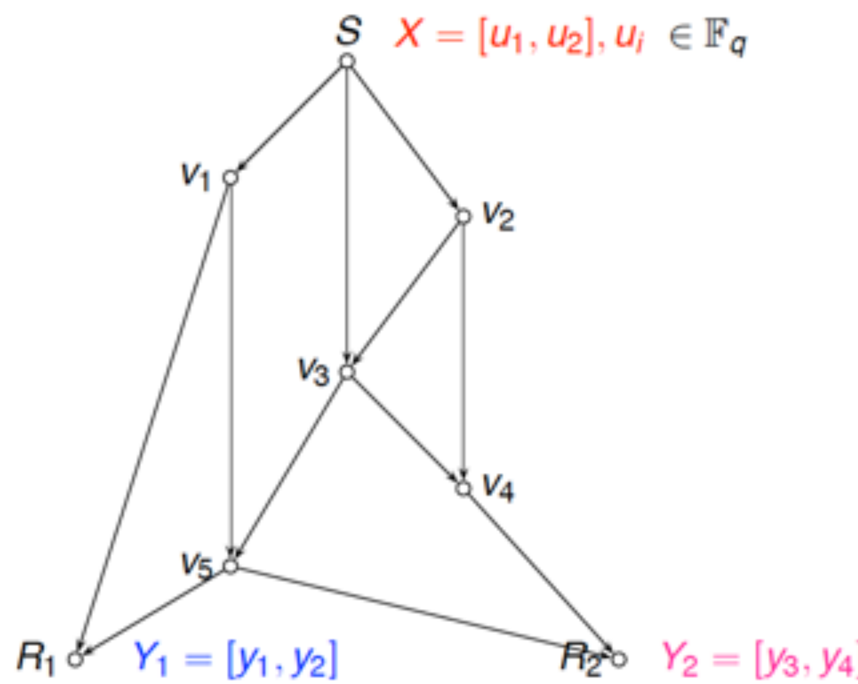
$$M = (u_1, u_2, \ldots, u_h) \in \mathbb{F}_q^h$$



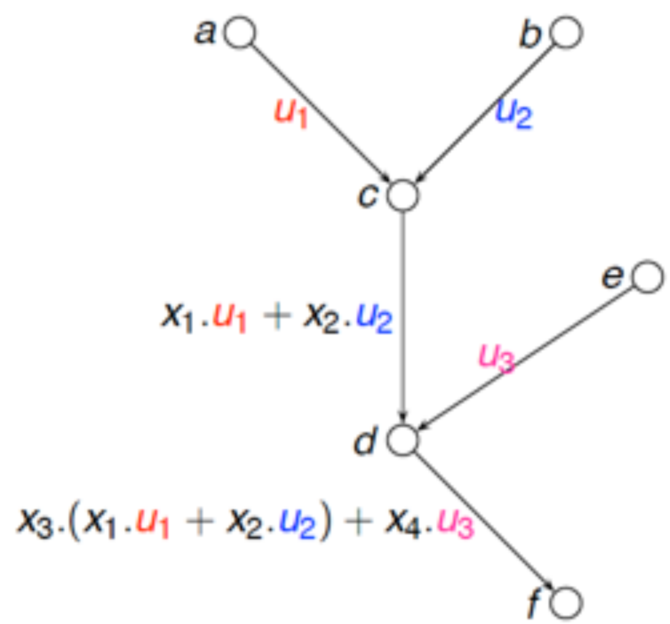
Algebraic Framework for Network Coding

(R. Koetter, M. Medard 2003) Linear scalar network code over sufficiently large finite field for multicast networks exists.

Multicast Network:

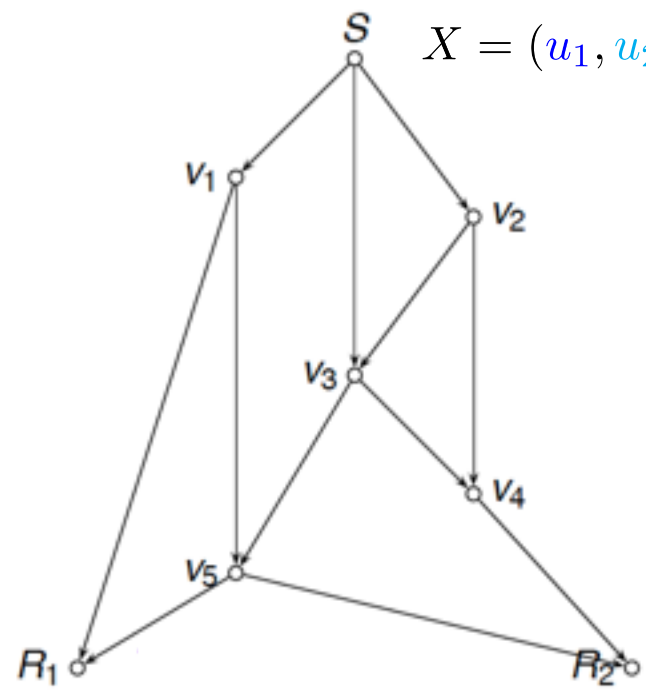


Linear Node Operation:



Transfer Matrix:

$X = (u_1, u_2)$



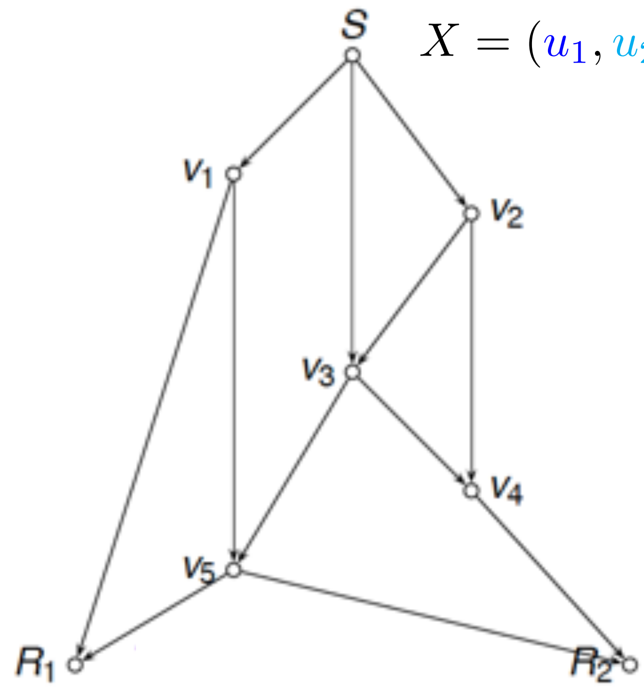
1

$Y_1 = (v_1, v_2)$ $Y_2 = (v_3, v_4)$

Transfer Matrix:

$X = (u_1, u_2)$

$Y_i = X.\mathbf{T}_i$



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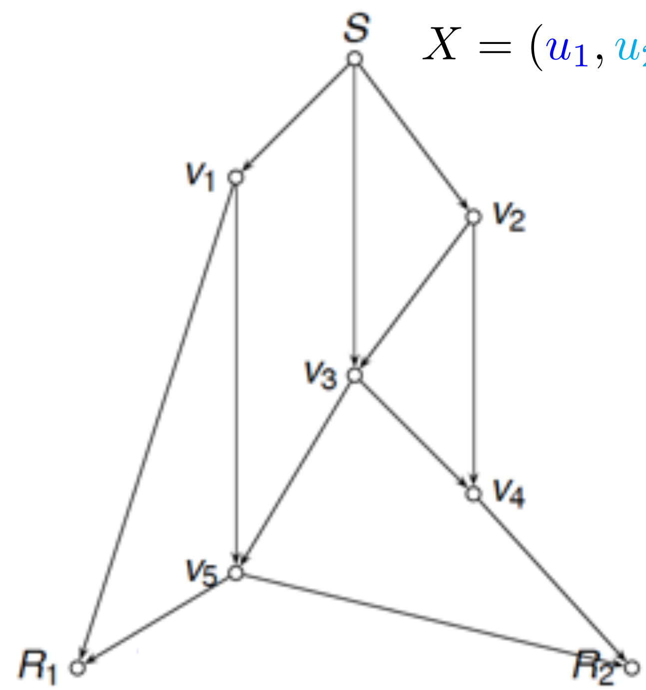

$X = (u_1, u_2)$

$Y_i = X.\mathbf{T}_i$

$$\mathbf{T}_1 = \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix}$$

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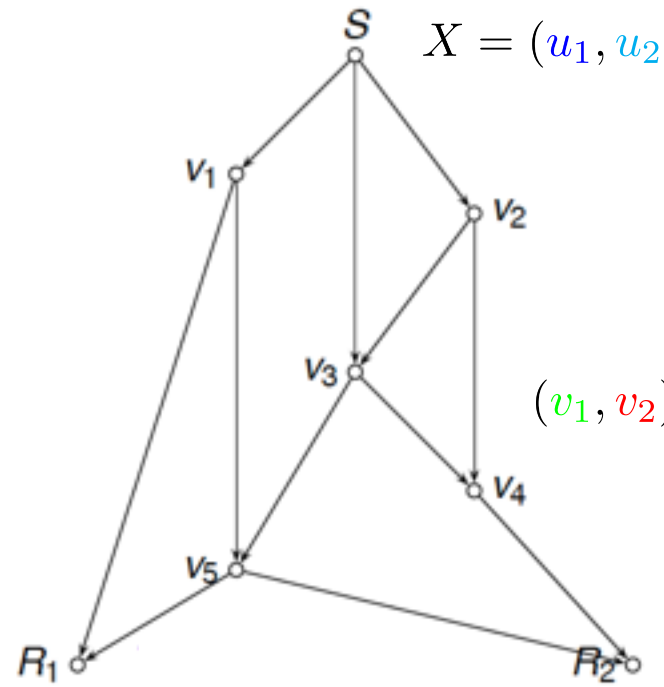

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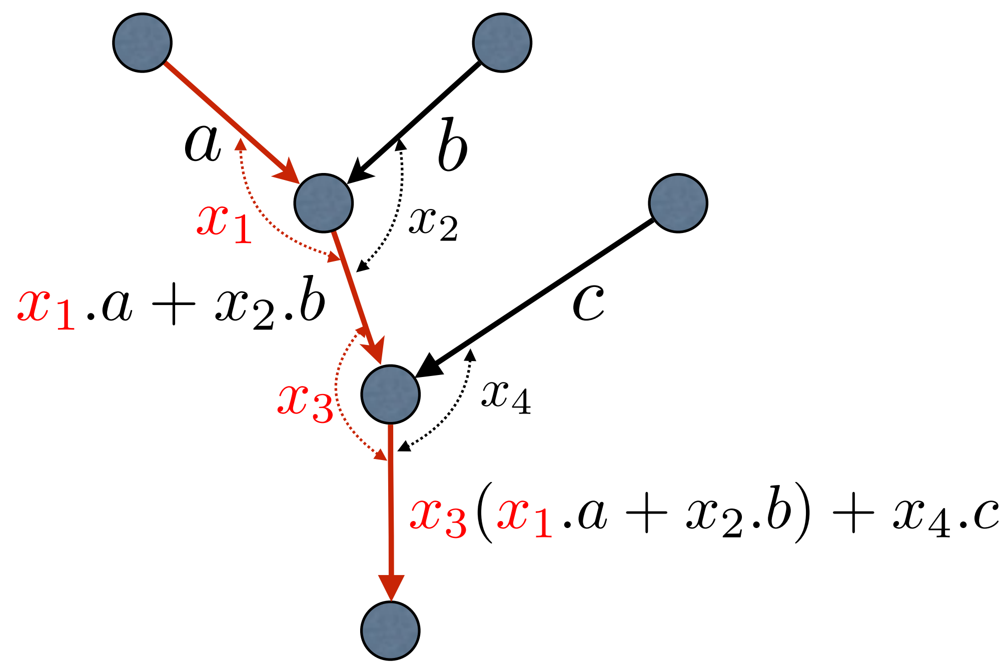
$$(v_1, v_2) = (u_1, u_2).\begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix}$$

$$Y_1 = (v_1, v_2) \qquad Y_2 = (v_3, v_4)$$

Linear Node Operation:



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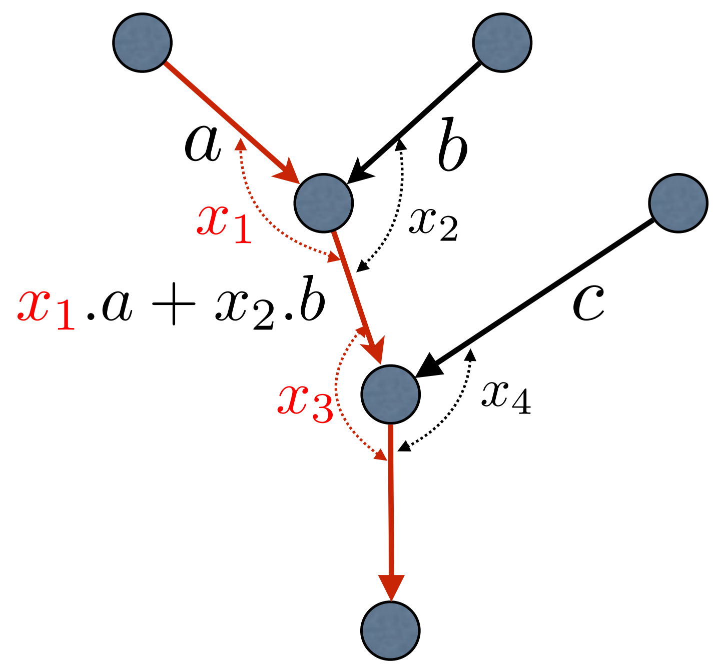

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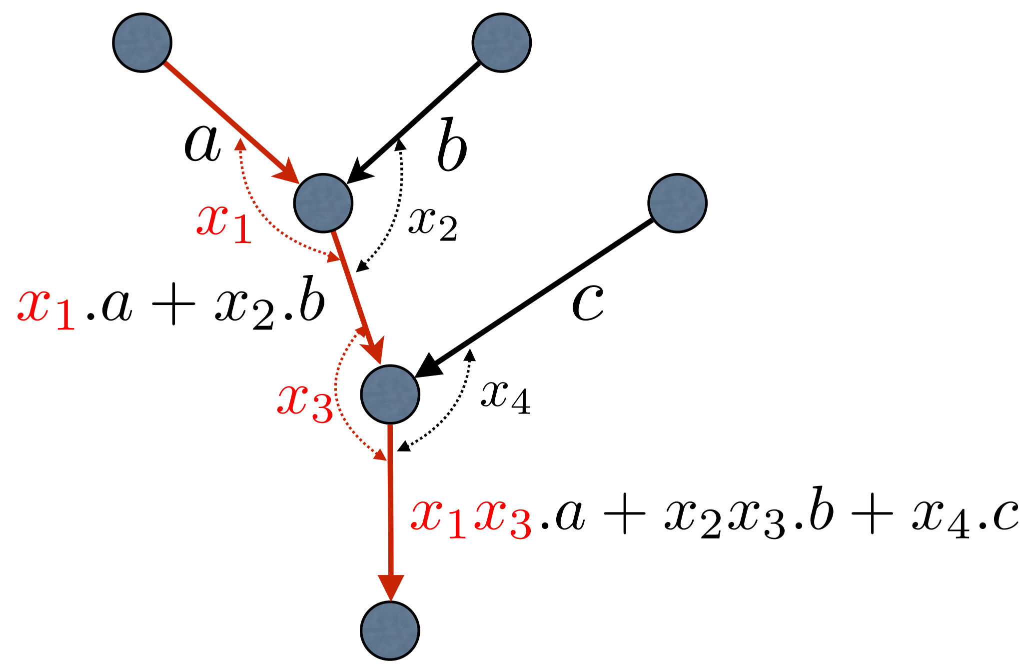
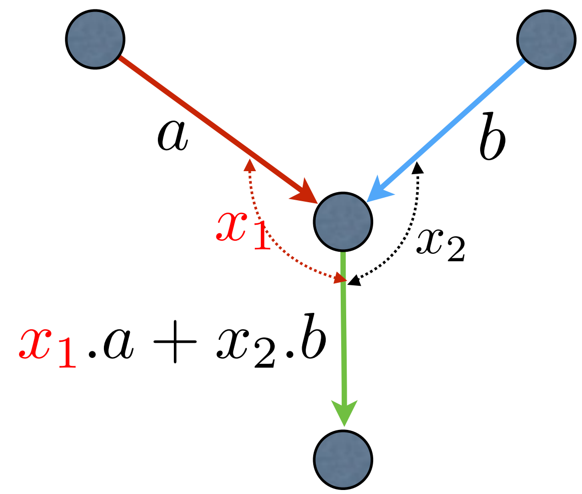


Networks and Line Networks:



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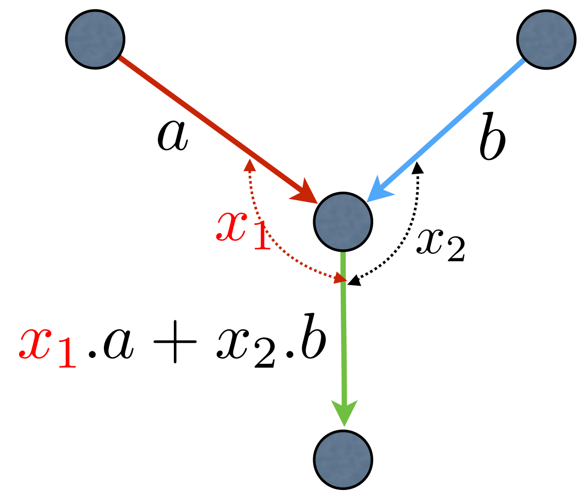

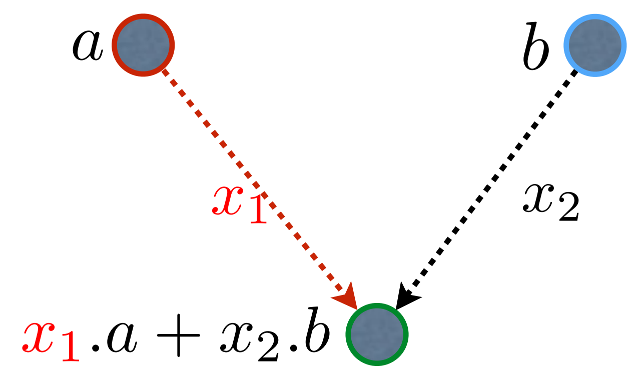

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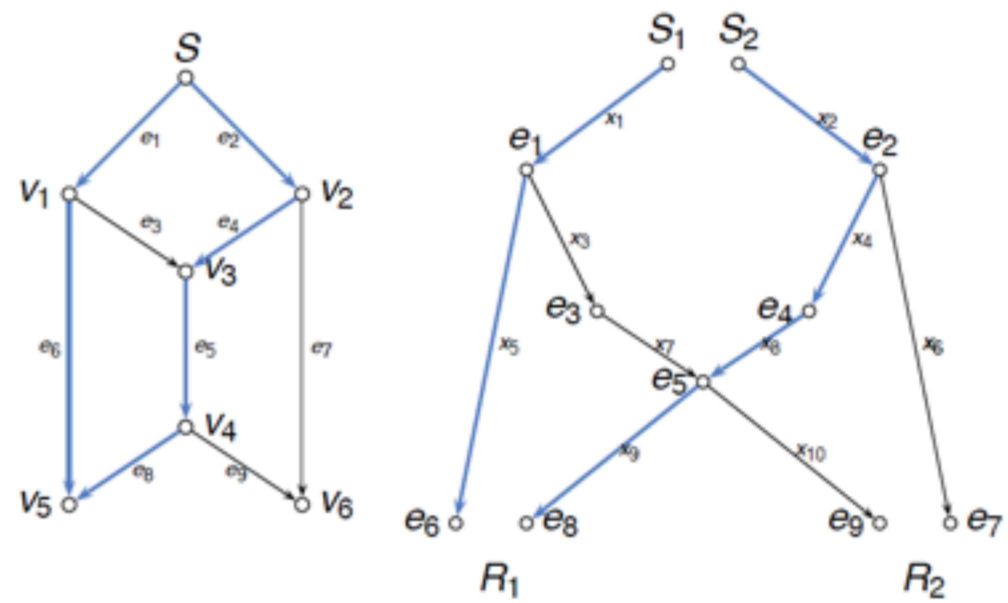
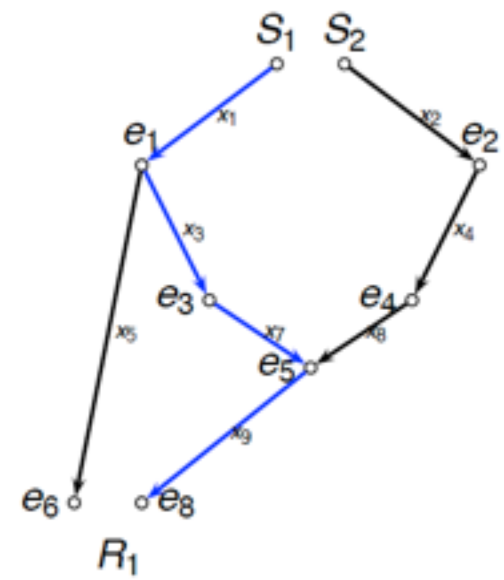


Transfer Matrix:



$$\mathbf{T}_1 = \begin{pmatrix} x_1 x_5 & x_1 x_3 x_7 x_9 \\ 0 & x_2 x_4 x_8 x_9 \end{pmatrix}$$

Terminology:

Concept	Description
Transfer matrix	Describes the relationship between the source and the received message
Overall transfer matrix	Product of the transfer matrices
Transfer polynomial	Determinant of the transfer matrix
Network polynomial	Determinant of the overall transfer matrix
Coding coefficients	Coefficients of the linear combinations of the messages at the relay nodes

Linear Network Code Design:

- X can be recovered from each Y_i.
- $Y_i = X.\mathbf{T_i}$ can be solved for X.
- $\mathbf{T_i}$ are invertible.
- Transfer polynomials are nonzero(for appropriate choice of coding coefficients).
- Network polynomial is nonzero(for appropriate choice of coding coefficients).

Motivation:

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- Network polynomials are directly related to network code design. (Koetter, Medard 2002, E. , Fragouli 2010, etc.)

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- Network polynomials are directly related to network code design. (Koetter, Medard 2002, E. , Fragouli 2010, etc.)
- Network polynomials are closely related to the field size optimisation problem.
- Network polynomials have applications in other traffic scenarios.
- It is an interesting theoretical problem.

Network Polynomial Application:

Lemma(E. , Fragouli): Network codes over a field $\mathbb{F}_{p^l}$ exists iff network polynomial does not belong to some "ideal".

Problem Formulation:

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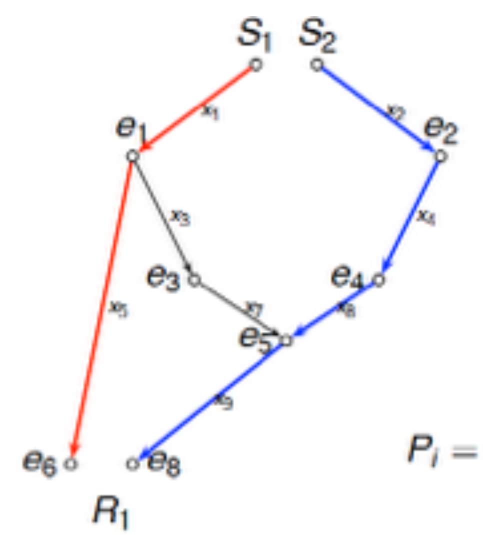

$$P_i = \det(\mathbf{T}_1) = x_1 x_5 x_2 x_4 x_8 x_9$$

Problem Formulation:

Monomials of network polynomials

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Subgraphs of line network

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Monomials of network polynomials

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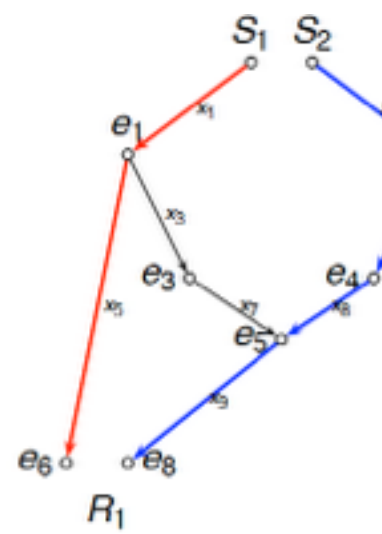
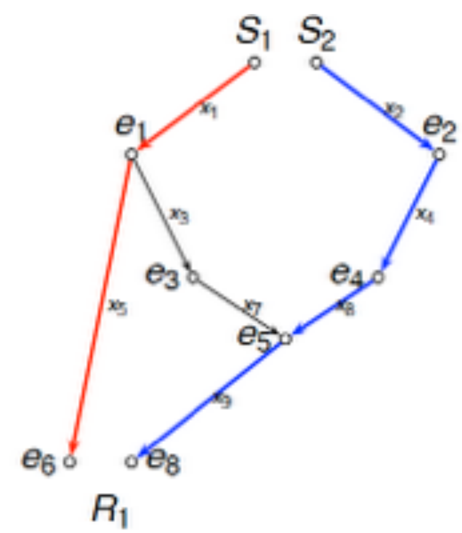
Subgraphs of line network



$P_i = \det(\mathbf{T}_1) = x_1 x_5 x_2 x_4 x_8 x_9$

Transfer Polynomial:



$$\mathbf{T}_1 = \begin{pmatrix} x_1 x_5 & x_1 x_3 x_7 x_9 \\ 0 & x_2 x_4 x_8 x_9 \end{pmatrix}, \qquad P_i = \det(\mathbf{T}_1) = x_1 x_5 x_2 x_4 x_8 x_9$$

Unicast:

Monomials of the transfer polynomial

↓

Union of *h* source-receiver paths in the line network

Unicast:

Monomials of the transfer polynomial

$\downharpoonleft\upharpoonright$?

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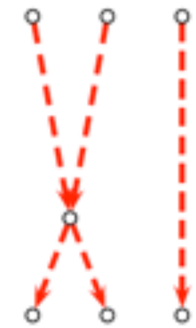
Monomials of transfer polynomial $=$

$+$

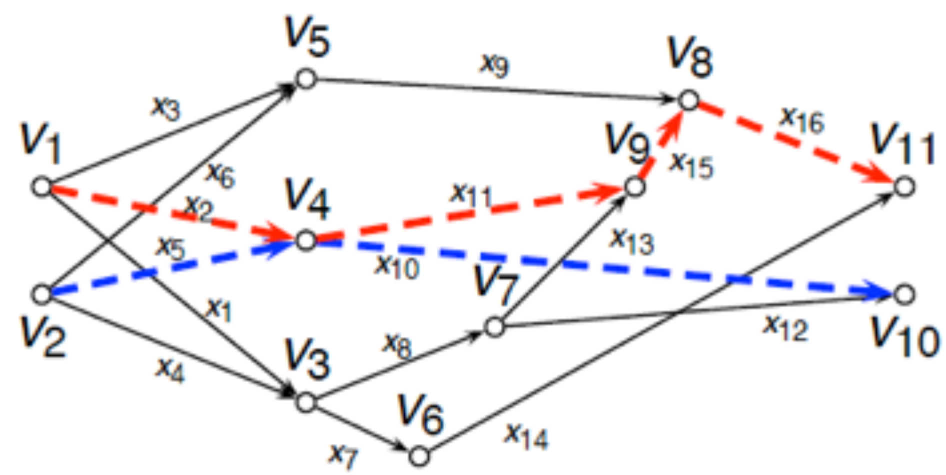


No cancellation

Full cancellation

Lindstorm, Gessel, Viennot (1989), Ebrahimi, Fragouli(2012).

Proof Idea:



v_5
x_9
v_8
x_3
v_1
v_9
x_{16}
v_{11}
x_6
x_{15}
v_4
x_2
x_{11}
x_5
x_{13}
x_{10}
v_7
x_1
v_2
x_{12}
v_{10}
x_4
v_3
x_8
v_6
x_{14}
x_7

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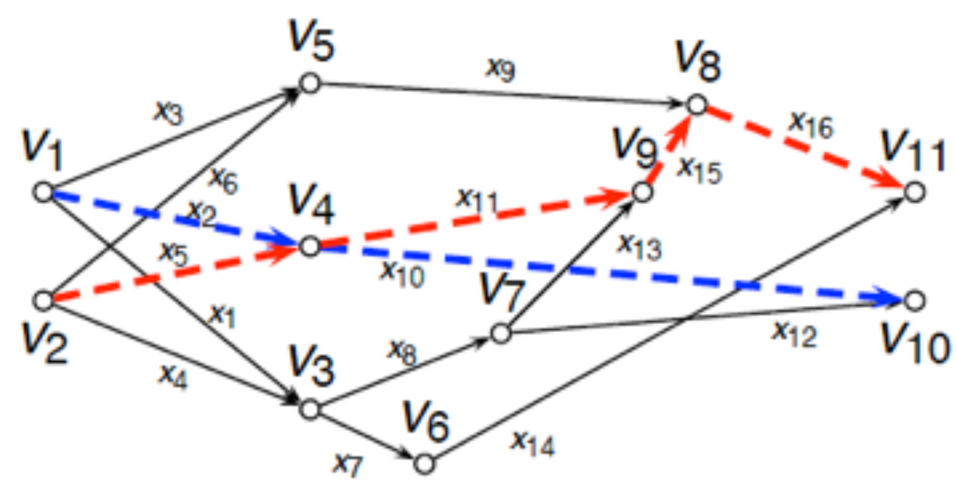

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v_7
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Multicast:

Network polynomial $P = \prod($ Transfer polynomials $P_i)$
$\Rightarrow$

Monomials of the network polynomial

$\Downarrow\Uparrow$?

Union of routings to each receiver.

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Union of routings to each receiver.

Two Receivers:

(E. , Fragouli 2012) For the networks with one or two receivers the full characterisation of the monomials of the network polynomial is proposed.

Proof Steps:

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1. If a monomial appears more than once, it will cancel out completely.

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2. Subgraphs that are uniquely decomposable into edge disjoint set of paths are characterised.

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Two Receivers:

Corollary 1: In the networks with 2 receivers, monomials have different subsets of variables.

Corollary 2: In those networks, there exists a network code over binary field.

Proof of Corollary 2:

$$p = x_1^2 x_2 x_5 x_7^2 + x_6^2 x_7^2 x_9^2 + x_3 x_4^2 + x_2^2 x_3^2 x_8^2 x_9 + x_1 x_4^2 x_5^2 x_6^2 x_8$$

Proof of Corollary 2:

$$p = x_1^2 x_2 x_5 x_7^2 + x_6^2 x_7^2 x_9^2 + x_3 x_4^2 + x_2^2 x_3^2 x_8^2 x_9 + x_1 x_4^2 x_5^2 x_6^2 x_8$$

$$x_3 = x_4 = 1$$

$$x_1 = x_2 = x_5 = x_6 = x_7 = x_8 = x_9 = 0$$

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$$p = x_1^2 x_2 x_5 x_7^2 + x_6^2 x_7^2 x_9^2 + x_3 x_4^2 + x_2^2 x_3^2 x_8^2 x_9 + x_1 x_4^2 x_5^2 x_6^2 x_8$$

$$x_3 = x_4 = 1$$
$$x_1 = x_2 = x_5 = x_6 = x_7 = x_8 = x_9 = 0$$

All but one monomial vanish!

Open Problem:

Question: For networks with arbitrary number of receivers, which subgraphs correspond to the monomials of the network polynomial?

Thank You!

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