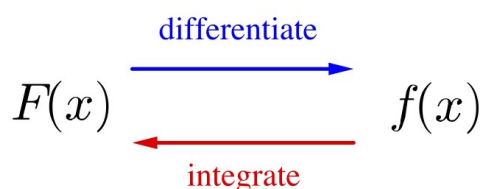


MATH 1510 Chapter 6

6.1 Indefinite integral

Integration is nothing but the reverse of differentiation.



To be more precise,

Definition 6.1. We call $F(x)$ an **antiderivative** of $f(x)$ if:

$$\frac{d}{dx}F(x) = f(x).$$

The collection of all antiderivatives of $f(x)$ is denoted by:

$$\int f(x) dx,$$

also called the **indefinite integral** of $f(x)$.

(For now, “ dx ” would just be part of the notation.)

Example 6.2. Since:

$$\frac{d}{dx} \left(\frac{1}{2}x^2 \right) = x,$$

the function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$.

Notice that we also have:

$$\frac{d}{dx} \left(\frac{1}{2}x^2 \right) = \frac{d}{dx} \left(\frac{1}{2}x^2 + 1 \right) = \frac{d}{dx} \left(\frac{1}{2}x^2 - \pi \right) = x.$$

Hence, the expressions:

$$\frac{1}{2}x^2, \quad \frac{1}{2}x^2 + 1, \quad \frac{1}{2}x^2 - \pi$$

all give antiderivatives of $f(x) = x$.

In fact, a function is an antiderivative of $f(x) = x$ if and only if it is equal to $\frac{1}{2}x^2 + C$ for some constant $C \in \mathbb{R}$.

Hence, we may represent the collection of all antiderivatives of $f(x) = x$ as follows:

$$\int x \, dx = \frac{1}{2}x^2 + C,$$

where C is an arbitrary constant.

Proposition 6.3. For any constants $a, b, k \in \mathbb{R}$,

$$\int (af(x) + bg(x)) \, dx = a \int f(x) \, dx + b \int g(x) \, dx;$$

$$\int x^k \, dx = \frac{1}{k+1}x^{k+1} + C \quad \text{if } k \neq -1;$$

$$\int x^{-1} \, dx = \ln |x| + C;$$

$$\int \sin x \, dx = -\cos x + C;$$

$$\int \cos x \, dx = \sin x + C;$$

$$\int e^x \, dx = e^x + C;$$

$$\int a^x \, dx = \frac{1}{\ln a}a^x + C;$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C;$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C.$$

Proof of Proposition 6.3. When $x > 0$, we have

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

On the other hand, if $x < 0$, we have

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$$

as desired. The other identities are just direct consequences of differentiation. $\square$

Example 6.4. •

$$\int \left(\cos x + \frac{2}{x} - 3^x \right) dx;$$

•

$$\int \frac{x^4}{1+x^2} dx.$$

6.2 Integration by Substitution

Sometimes, integration can be handled by “change of variable”:

Theorem 6.5 (Integration by Substitution). *Assuming differentiability and integrability, suppose*

$$y = g(x) \quad \text{and} \quad f(x) = h(y) \frac{dy}{dx}.$$

Then,

$$\int f(x) dx = \int h(y) dy$$

Proof of Integration by Substitution. Let $H(y)$ be the antiderivative of $h(y)$. Then,

$$\frac{d}{dx} H(g(x)) = H'(g(x)) g'(x) = h(g(x)) g'(x) = h(y) \frac{dy}{dx} = f(x)$$

Hence,

$$\int f(x) dx = H(g(x)) + C = H(y) + C = \int h(y) dy$$

$\square$

The formula is easier to remember if we use notations like $\left(\frac{dy}{dx}\right) dx = dy$, in which the part “ dx ” becomes crucial.

Example 6.6. To evaluate

$$\int e^x \sin(e^x) dx,$$

we let $u = e^x$. By the fact that

$$\frac{du}{dx} = e^x \implies du = e^x dx$$

we have

$$\begin{aligned} \int e^x \sin(e^x) dx &= \int \sin(e^x)(e^x dx) \\ &= \int \sin u du \\ &= -\cos u + C \\ &= -\cos(e^x) + C. \end{aligned}$$

Remember to change everything into u .

Example 6.7. • Evaluate

$$\int \frac{e^{\sqrt{x}} \cos(e^{\sqrt{x}})}{\sqrt{x}} dx$$

by the substitution $u = e^{\sqrt{x}}$.

• Evaluate

$$\int x \sin(x^2) dx.$$

Example 6.8. Sometimes, if it's not too complicated, it might be preferable to do the substitution without introducing a new variable:

$$\begin{aligned} \int \frac{1}{x-2} dx &= \int \frac{1}{x-2} d(x-2) \quad (\text{because } \frac{d(x-2)}{dx} = 1) \\ &= \ln|x-2| + C. \\ \int \frac{x}{x^2+1} dx &= \int \frac{1}{x^2+1} d\left(\frac{1}{2}x^2\right) \quad (\text{because } \frac{d(\frac{1}{2}x^2)}{dx} = x) \\ &= \frac{1}{2} \int \frac{1}{x^2+1} d(x^2+1) \\ &= \frac{1}{2} \ln|x^2+1| + C. \end{aligned}$$

6.3 Integrating $\sin^m x \cos^n x$

To evaluate:

$$\int \sin^m x \cos^n x dx,$$

where m, n are non-negative integers, we consider three different cases.

Case 1: m is odd ($m = 2p + 1$) In this case, we use the substitution $u = \cos x$:

$$\begin{aligned} \int \sin^m x \cos^n x dx &= \int \sin^{2p+1} x \cos^n x dx \\ &= - \int \sin^{2p} x \cos^n x d(\cos x) \\ &= - \int (1 - \cos^2 x)^p \cos^n x d(\cos x) \\ &= - \int (1 - u^2)^p u^n du \end{aligned}$$

Case 2: n is odd ($n = 2q + 1$) In this case, we use the substitution $u = \sin x$:

$$\begin{aligned} \int \sin^m x \cos^n x dx &= \int \sin^m x \cos^{2q+1} x dx \\ &= \int \sin^m x \cos^{2q} x d(\sin x) \\ &= \int \sin^m x (1 - \sin^2 x)^q d(\sin x) \\ &= \int u^m (1 - u^2)^q du \end{aligned}$$

Case 3: m, n are both even ($m = 2p, n = 2q$) In this case, we use half-angle formulas to reduce the powers:

$$\begin{aligned} \int \sin^m x \cos^n x dx &= \int \sin^{2p} x \cos^{2q} x dx \\ &= \int (\sin^2 x)^p (\cos^2 x)^q dx \\ &= \int \left(\frac{1}{2}(1 - \cos 2x) \right)^p \left(\frac{1}{2}(1 + \cos 2x) \right)^q dx \\ &= \frac{1}{2^{p+q}} \int (1 - \cos 2x)^p (1 + \cos 2x)^q dx. \end{aligned}$$

(Notice that the integrand is a polynomial of $\cos 2x$ with degree $p+q = \frac{1}{2}(m+n)$.)

Example 6.9. •

$$\int \sin^2 x \, dx$$

•

$$\int \cos^3 x \, dx$$

•

$$\int \sin^5 x \, dx$$

6.4 Integration by Trigonometric Substitution

The idea is as follows:

- Use the substitution $x = a \sin t$ when “ $a^2 - x^2$ ” occurs (because $1 - \sin^2 t = \cos^2 t$).
- Use the substitution $x = a \tan t$ when “ $a^2 + x^2$ ” occurs (because $1 + \tan^2 t = \sec^2 t$).
- Use the substitution $x = a \sec t$ when “ $x^2 - a^2$ ” occurs (because $\sec^2 t - 1 = \tan^2 t$).

Example 6.10.

$$\begin{aligned} \int \sqrt{2-x^2} \, dx &= \int \sqrt{2 - (\sqrt{2} \sin t)^2} d(\sqrt{2} \sin t) \quad (\text{by letting } x = \sqrt{2} \sin t) \\ &= \int \sqrt{2} \cos t (\sqrt{2} \cos t \, dt) \\ &= \int 2 \cos^2 t \, dt \\ &= \int (\cos 2t + 1) \, dt \\ &= \frac{1}{2} \sin 2t + t + C \\ &= \frac{1}{2} \sin \left(2 \arcsin \left(\frac{x}{\sqrt{2}} \right) \right) + \arcsin \left(\frac{x}{\sqrt{2}} \right) + C \\ &= \frac{1}{2} x \sqrt{2-x^2} + \arcsin \left(\frac{x}{\sqrt{2}} \right) + C. \end{aligned}$$

(In this course, when handling indefinite integrals, we usually assume t lies in an appropriate region so that $\sqrt{\cos^2 t} = \cos t$, etc., for simplicity.)

Example 6.11.

$$\int \frac{1}{(x^2 + x + 1)^2} dx$$

6.5 Integration by Partial Fraction

How to evaluate:

$$\int \frac{1}{x^2 - 1} dx?$$

By **partial fraction decomposition**, we have:

$$\frac{1}{x^2 - 1} = \frac{1}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

for some constants A, B . By some simple computations, we know that $A = \frac{1}{2}, B = -\frac{1}{2}$. Hence,

$$\begin{aligned} \int \frac{1}{x^2 - 1} dx &= \int \left(\frac{\frac{1}{2}}{x - 1} + \frac{-\frac{1}{2}}{x + 1} \right) dx \\ &= \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C. \end{aligned}$$

In general, if $\deg P(x) < \deg Q(x)$, by factorizing $Q(x)$ into linear or *irreducible* quadratic terms (with real coefficients), we have:

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{a(x - b_1)^{n_1}(x - b_2)^{n_2} \cdots (x^2 + c_1x + d_1)^{m_1}(x^2 + c_2x + d_2)^{m_2} \cdots}$$

Then, by partial fraction decomposition, we have:

$$\begin{aligned}
& \frac{P(x)}{a(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x^2+c_1x+d_1)^{m_1}(x^2+c_2x+d_2)^{m_2}\cdots} \\
&= \frac{\frac{*}{(x-b_1)^{n_1}} + \frac{*}{(x-b_1)^{n_1-1}} + \cdots + \frac{*}{(x-b_1)^1}}{a(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x^2+c_1x+d_1)^{m_1}(x^2+c_2x+d_2)^{m_2}\cdots} \\
&+ \frac{\frac{*}{(x-b_2)^{n_2}} + \frac{*}{(x-b_2)^{n_2-1}} + \cdots + \frac{*}{(x-b_2)^1}}{a(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x^2+c_1x+d_1)^{m_1}(x^2+c_2x+d_2)^{m_2}\cdots} \\
&+ \cdots (\text{similarly for the other linear terms}) \\
&+ \frac{\frac{*x+*}{(x^2+c_1x+d_1)^{m_1}} + \frac{*x+*}{(x^2+c_1x+d_1)^{m_1-1}} + \cdots + \frac{*x+*}{(x^2+c_1x+d_1)^1}}{a(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x^2+c_1x+d_1)^{m_1}(x^2+c_2x+d_2)^{m_2}\cdots} \\
&+ \frac{\frac{*x+*}{(x^2+c_2x+d_2)^{m_2}} + \frac{*x+*}{(x^2+c_2x+d_2)^{m_2-1}} + \cdots + \frac{*x+*}{(x^2+c_2x+d_2)^1}}{a(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x^2+c_1x+d_1)^{m_1}(x^2+c_2x+d_2)^{m_2}\cdots} \\
&+ \cdots (\text{similarly for the other quadratic terms})
\end{aligned}$$

where $*$ stands for some undetermined constants.

If $\deg P(x) \geq \deg Q(x)$, we may apply long division first to obtain:

$$\frac{P(x)}{Q(x)} = A(x) + \frac{B(x)}{Q(x)}$$

where $\deg B(x) < \deg Q(x)$, and then apply partial fraction decomposition to $\frac{B(x)}{Q(x)}$.

Example 6.12. To evaluate

$$\int \frac{x^3 + 6x + 1}{(x^2 - 1)^2(x^2 + 1)} dx,$$

we apply partial fraction decomposition:

$$\begin{aligned}
\frac{x^3 + 6x + 1}{(x^2 - 1)^2(x^2 + 1)} &= \frac{x^3 + 6x + 1}{(x - 1)^2(x + 1)^2(x^2 + 1)} \\
&= \frac{A}{(x - 1)^2} + \frac{B}{x - 1} + \frac{C}{(x + 1)^2} + \frac{D}{x + 1} + \frac{Ex + F}{x^2 + 1}
\end{aligned}$$

for some constants A, B, C, D, E, F . By some tedious computations,

$$A = 1, B = -\frac{7}{8}, C = -\frac{3}{4}, D = -\frac{3}{8}, E = \frac{5}{4}, F = \frac{1}{4}.$$

Moreover,

$$\begin{aligned}
\int \frac{1}{x \pm 1} dx &= \ln |x \pm 1| + C', \\
\int \frac{1}{(x \pm 1)^2} dx &= \int (x \pm 1)^{-2} dx = -(x \pm 1)^{-1} + C', \\
\int \frac{1}{x^2 + 1} dx &= \arctan x + C', \\
\int \frac{x}{x^2 + 1} dx &= \int \frac{1}{x^2 + 1} d\left(\frac{1}{2}x^2\right) = \frac{1}{2} \int \frac{1}{x^2 + 1} d(x^2 + 1) \\
&= \frac{1}{2} \ln |x^2 + 1| + C'.
\end{aligned}$$

Hence, we can conclude that

$$\begin{aligned}
&\int \frac{x^3 + 6x + 1}{(x^2 - 1)^2(x^2 + 1)} dx \\
&= -A(x - 1)^{-1} + B \ln |x - 1| - C(x + 1)^{-1} + D \ln |x + 1| + \frac{E}{2} \ln |x^2 + 1| + F \arctan x + C'.
\end{aligned}$$

Example 6.13.

$$\int \frac{x^3}{x^2 - 1} dx$$

By our procedure, we can integrate any rational function as long as we can integrate the building blocks:

$$\frac{1}{x - b}, \frac{1}{(x - b)^n}, \frac{x}{x^2 + cx + d}, \frac{1}{x^2 + cx + d}, \frac{x}{(x^2 + cx + d)^m}, \frac{1}{(x^2 + cx + d)^m}$$

where $n, m \geq 2$. We have handled the first four in the above example (when $c = 0, d = 1$). For general c, d , we need to complete the square.

Example 6.14.

$$\begin{aligned}
\int \frac{x + 2}{x^2 + 2x + 2} dx &= \int \frac{x + 2}{(x + 1)^2 + 1} dx \\
&= \int \frac{u + 1}{u^2 + 1} du \quad (\text{by letting } u = x + 1) \\
&= \int \frac{u}{u^2 + 1} du + \int \frac{1}{u^2 + 1} du \\
&= \frac{1}{2} \ln |u^2 + 1| + \arctan u + C \\
&= \frac{1}{2} \ln |(x + 1)^2 + 1| + \arctan(x + 1) + C.
\end{aligned}$$

Finally, when $m \geq 1$, we can still complete the square to force the form $c = 0, d = 1$.

Example 6.15. By trigonometric substitution, evaluate

$$\int \frac{1}{(x^2 + 1)^2} dx.$$

6.6 Integration by Rationalization

Example 6.16. To evaluate

$$\int \frac{\sqrt{x}}{x+1} dx,$$

we let $u = \sqrt{x}$. Since

$$u = \sqrt{x} \implies x = u^2 \implies dx = 2u du,$$

$$\begin{aligned} \int \frac{\sqrt{x}}{x+1} dx &= \int \frac{u}{u^2+1} (2u du) \\ &= \int \frac{2u^2}{u^2+1} du \quad (\text{which is a rational function now}) \\ &= 2 \int \left(1 - \frac{1}{u^2+1} \right) du \\ &= 2u - 2 \arctan u + C \\ &= 2\sqrt{x} - 2 \arctan \sqrt{x} + C. \end{aligned}$$

Example 6.17. To evaluate

$$\int \frac{\sqrt{x}}{\sqrt[3]{x}+1} dx,$$

with some thoughts, it's not hard to see that we should use $u = x^{\frac{1}{6}}$. Then, similarly,

$$u = x^{\frac{1}{6}} \implies x = u^6 \implies dx = 6u^5 du.$$

Hence,

$$\begin{aligned}
\int \frac{\sqrt{x}}{\sqrt[3]{x}+1} dx &= \int \frac{6u^8}{u^2+1} du \\
&= 6 \int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1} \right) du \\
&= 6 \left(\frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \arctan u \right) + C \\
&= 6 \left(\frac{1}{7}x^{\frac{7}{6}} - \frac{1}{5}x^{\frac{5}{6}} + \frac{1}{3}x^{\frac{3}{6}} - x^{\frac{1}{6}} + \arctan(x^{\frac{1}{6}}) \right) + C.
\end{aligned}$$

6.7 Integrating Basic Trigonometric Functions

Proposition 6.18.

$$\begin{aligned}
\int \sin x \, dx &= -\cos x + C \\
\int \cos x \, dx &= \sin x + C \\
\int \tan x \, dx &= \ln |\sec x| + C
\end{aligned}$$

$$\begin{aligned}
\int \sec x \, dx &= \ln |\sec x + \tan x| + C \\
\int \csc x \, dx &= -\ln |\csc x + \cot x| + C \\
\int \cot x \, dx &= -\ln |\csc x| + C
\end{aligned}$$

Proof of Proposition 6.18.

$$\begin{aligned}
\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\
&= \int \frac{1}{\cos x} \, d(-\cos x) \\
&= -\ln |\cos x| + C \\
&= \ln |\sec x| + C.
\end{aligned}$$

$\cot x$ can be handled similarly.

$$\begin{aligned}
 \int \sec x \, dx &= \int \frac{\cos x}{\cos^2 x} \, dx \\
 &= \int \frac{1}{1 - \sin^2 x} \, d(\sin x) \\
 &= \int \frac{1}{1 - u^2} \, du \quad (\text{by letting } u = \sin x) \\
 &= \frac{1}{2} \int \left(\frac{1}{1 - u} + \frac{1}{1 + u} \right) \, du \\
 &= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C \\
 &= \ln \left| \frac{1 + \sin x}{\cos x} \right| + C.
 \end{aligned}$$

$\csc x$ can be handled similarly. □

6.8 Integration by t -Substitution

Suppose we want to evaluate

$$\int \frac{1}{1 + \sin x} \, dx.$$

If we let $t = \tan \frac{x}{2}$, then

$$\begin{aligned}
 \sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} = \frac{2t}{1 + t^2} \\
 \cos x &= \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos^2 \frac{x}{2} (1 - t^2) = \frac{1 - t^2}{1 + t^2}
 \end{aligned}$$

Furthermore,

$$x = 2 \arctan t \implies dx = \frac{2}{1 + t^2} \, dt.$$

Hence,

$$\begin{aligned}
 \int \frac{1}{1 + \sin x} \, dx &= \int \frac{1}{1 + \frac{2t}{1+t^2}} \cdot \frac{2}{1 + t^2} \, dt \\
 &= \int \frac{2}{(t + 1)^2} \, dt \\
 &= -2(t + 1)^{-1} + C \\
 &= -2 \left(\tan \frac{x}{2} + 1 \right)^{-1} + C.
 \end{aligned}$$

Theorem 6.19 (t-Substitution). *By letting $t = \tan \frac{x}{2}$, we have*

$$dx = \frac{2}{1+t^2} dt,$$

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}.$$

With t -substitution, we can transform any rational functions of trigonometric functions into an algebraic rational functions, which can then be handled by partial fractions.

Example 6.20.

$$\int \frac{1}{\sin x + 2 \cos x + 1} dx.$$

6.9 Integration by Parts

Theorem 6.21 (Integration by Parts).

$$\int u dv = uv - \int v du.$$

Proof of Integration by Parts. From product rule,

$$\begin{aligned} \frac{d}{dx}(uv) &= v \frac{du}{dx} + u \frac{dv}{dx} \\ \int \frac{d}{dx}(uv) dx &= \int v \frac{du}{dx} dx + \int u \frac{dv}{dx} dx \\ uv + C &= \int v du + \int u dv \\ \int u dv &= uv - \int v du \end{aligned}$$

(C can be omitted because of the remained indefinite integrals)

□

Example 6.22. To evaluate

$$\int x e^x dx,$$

we let $u = x$ and $v = e^x$. Then, $dv = e^x dx$ and we have

$$\begin{aligned}\int x e^x dx &= \int x d(e^x) \\ &= x e^x - \int e^x dx \\ &= x e^x - e^x + C.\end{aligned}$$

Example 6.23. To evaluate:

$$\int x^2 \cos x dx,$$

we let $u = x^2$ and $v = \sin x$. Then, $dv = \cos x dx$ and we have

$$\begin{aligned}\int x^2 \cos x dx &= \int x^2 d(\sin x) \\ &= x^2 \sin x - \int \sin x d(x^2) \\ &= x^2 \sin x - \int 2x \sin x dx.\end{aligned}$$

So, to apply integration by parts once, we practically integrate $\cos x$ once and differentiate x^2 once. That's how to determine which function to be u and which function to be v . We then proceed to apply integration by parts one more time:

$$\begin{aligned}\int x^2 \cos x dx &= x^2 \sin x - \int 2x \sin x dx \\ &= x^2 \sin x + 2 \int x d(\cos x) \\ &= x^2 \sin x + 2x \cos x - 2 \int \cos x dx \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C.\end{aligned}$$

Example 6.24. How about

$$\int \arcsin x dx?$$

We don't know how to integrate $\arcsin x$, but we know how to differentiate it. So,

$$\begin{aligned}
 \int \arcsin x \, dx &= x \arcsin x - \int x d(\arcsin x) \\
 &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx \\
 &= x \arcsin x - \int \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} d(\sin \theta) \quad (\text{by letting } x = \sin \theta) \\
 &= x \arcsin x - \int \sin \theta \, d\theta \\
 &= x \arcsin x + \cos \theta + C \\
 &= x \arcsin x + \sqrt{1-x^2} + C.
 \end{aligned}$$

Example 6.25. •

$$\int \ln x \, dx$$

•

$$\int e^x \sin x \, dx$$

6.10 Reduction Formula

Example 6.26. How to evaluate:

$$\int x^4 e^x \, dx ?$$

Instead of applying integration by parts four times, we could set up a **reduction formula** as follows.

Let

$$I_n = \int x^n e^x \, dx,$$

where n is a non-negative integer. By integration by parts,

$$\begin{aligned}
 \int x^n e^x \, dx &= \int x^n d(e^x) \\
 &= x^n e^x - \int e^x d(x^n) \\
 &= x^n e^x - n \int x^{n-1} e^x \, dx
 \end{aligned}$$

provided that $n \geq 1$. In other words,

$$I_n = x^n e^x - n I_{n-1} \quad \text{for all } n \geq 1.$$

All we need now would be the initial result:

$$I_0 = \int x^0 e^x dx = \int e^x dx = e^x + C.$$

We can then easily generate I_n up to any n using our reduction formula:

$$\begin{aligned} I_1 &= x^1 e^x - I_0 = x e^x - e^x + C \\ I_2 &= x^2 e^x - 2I_1 = x^2 e^x - 2x e^x + 2e^x + C \\ I_3 &= x^3 e^x - 3I_2 = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\ I_4 &= x^4 e^x - 4I_3 = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + C. \end{aligned}$$

Example 6.27. Let:

$$I_n = \int \frac{1}{x^n(x+1)} dx,$$

where n is a non-negative integer. This problem can be solved by partial fraction decomposition if n is given. Interestingly, we can also set up a reduction formula as follows. For $n \geq 1$,

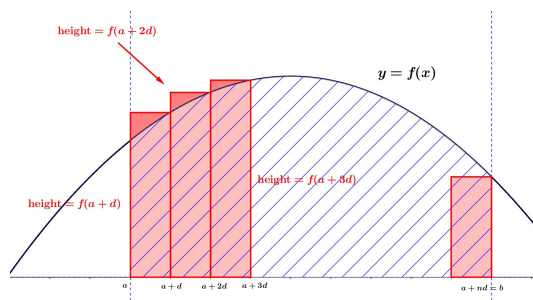
$$\begin{aligned} I_n &= \int \frac{1}{x^n(x+1)} dx \\ &= \int \left(\frac{1+x}{x^n(x+1)} - \frac{x}{x^n(x+1)} \right) dx \\ &= \int \frac{1}{x^n} dx - \int \frac{1}{x^{n-1}(x+1)} dx \\ &= \begin{cases} \ln|x| - I_0 & \text{if } n = 1; \\ \frac{1}{-n+1} x^{-n+1} - I_{n-1} & \text{if } n \geq 2. \end{cases} \end{aligned}$$

For $n = 0$, we have:

$$I_0 = \int \frac{1}{x+1} dx = \ln|x+1| + C.$$

6.11 Definite Integral and Riemann Sum

For a continuous function $f(x)$, how to find the area of the region under the curve $y = f(x)$ over the interval $[a, b]$?



If we cut the interval $[a, b]$ into n parts, then the width of each would be d . The right end-points of the sub-intervals will then be:

$$a + d, a + 2d, a + 3d, \dots, a + nd = b.$$

So, the heights of the rectangles in the above graph are

$$f(a + d), f(a + 2d), f(a + 3d), \dots, f(a + nd).$$

Therefore, $\sum_{k=1}^n f(a + kd)d$ is the total area of the rectangles. When n goes to infinity, this value will be exactly the (signed) area under the curve $y = f(x)$ over the interval $[a, b]$.

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Definition 6.28. The **definite integral** of a piecewise continuous function $f(x)$ over an interval $[a, b]$ is

$$\int_a^b f(x) dx = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(a + kd)d \quad \text{where } d = \frac{b-a}{n}$$

Proof of Definition 6.28. For its well-definedness, see Definition 1 and Theorem 1 in Appendix 5. $\square$

It will be convenient to extend our definition to arbitrary a, b :

$$\int_a^a f(x) dx = 0 \quad \text{and} \quad \int_a^b f(x) dx = - \int_b^a f(x) dx \quad \text{if } a > b.$$

Notice that if we let $x_k = a + kd$ be the right end-points of the sub-intervals, then

$$\Delta x_k = x_k - x_{k-1} = d$$

$$\int f(x) \, dx = \lim \sum f(x_k) \, \Delta x_k$$

small change of x
infinite sum

That explains why we keep “ dx ” as part of the notation.

Example 6.29. Consider the function $f(x) = x^2$ over the interval $[a, b]$. By definition,

$$\begin{aligned} \int_a^b f(x) \, dx &= \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(a + kd) d \quad \text{where } d = \frac{b-a}{n} \\ &= (b-a) \lim_{n \rightarrow +\infty} \frac{1}{n} ((a+d)^2 + (a+2d)^2 + \cdots + (a+nd)^2) \\ &= (b-a) \lim_{n \rightarrow +\infty} \frac{1}{n} ((a^2 + 2ad + d^2) + (a^2 + 4ad + 4d^2) + \cdots + (a^2 + 2and + n^2d^2)) \\ &= (b-a) \lim_{n \rightarrow +\infty} \frac{1}{n} \left(na^2 + 2ad \frac{n(n+1)}{2} + d^2 \frac{n(n+1)(2n+1)}{6} \right) \\ &= (b-a) \lim_{n \rightarrow +\infty} \frac{1}{n} \left(na^2 + 2a \frac{b-a}{n} \frac{n(n+1)}{2} + \frac{(b-a)^2}{n^2} \frac{n(n+1)(2n+1)}{6} \right) \\ &= (b-a) \left(a^2 + 2a(b-a) \frac{1}{2} + (b-a)^2 \frac{2}{6} \right) \\ &= \frac{1}{3} (b^3 - a^3). \end{aligned}$$

Proposition 6.30. For any constants $a, b, c, \alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\int_a^b f(x) dx &= - \int_b^a f(x) dx; \\ \int_a^b f(x) dx &= \int_a^c f(x) dx + \int_c^b f(x) dx; \\ \int_a^b (\alpha f(x) + \beta g(x)) dx &= \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx; \\ f(x) \leq g(x) \text{ on } [a, b] &\implies \int_a^b f(x) dx \leq \int_a^b g(x) dx.\end{aligned}$$

Proof of Proposition 6.30. See Proposition 2 in Appendix 5. □

6.12 Fundamental Theorem of Calculus

For a continuous function $f(t)$, we may define a function:

$$F(x) := \int_a^x f(t) dt$$

, where a is any element in the domain of $f(x)$.

We are now ready to state and prove the following fundamental results in Calculus, which basically mean

- “Integration and differentiation are reverse of each other.”
- “Definite integrals can be computed by indefinite integrals.”

Theorem 6.31 (Fundamental theorem of calculus FTC). • **Part I:** If $f(x)$ is a continuous function on $[a, b]$, then the function $F(x) = \int_a^x f(t)dt$ is differentiable on $[a, b]$ and:

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

for all $x \in [a, b]$.

- **Part II:** If $F(x)$ is a differentiable function on $[a, b]$ and $F'(x)$ is continuous on $[a, b]$, then:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Proof of Fundamental theorem of calculus (FTC). First: For any $x \in (a, b)$ and small $h > 0$, by EVT, we can define $m(h)$ and $M(h)$ such that f attains its minimum and maximum within $[x, x+h]$ respectively. Then,

$$RF'(x) = \lim_{h \rightarrow 0^+} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0^+} \frac{\int_a^{x+h} f(t) dt - \int_a^x f(t) dt}{h} = \lim_{h \rightarrow 0^+} \frac{\int_x^{x+h} f(t) dt}{h}$$

Since

$$f(m(h)) = \frac{\int_x^{x+h} f(m(h)) dt}{h} \leq \frac{\int_x^{x+h} f(t) dt}{h} \leq \frac{\int_x^{x+h} f(M(h)) dt}{h} = f(M(h))$$

and, by continuity,

$$\lim_{h \rightarrow 0^+} f(m(h)) = f(x) = \lim_{h \rightarrow 0^+} f(M(h)),$$

we can conclude that $RF'(x) = f(x)$ (squeeze theorem). Similarly,

$$LF'(x) = \lim_{h \rightarrow 0^-} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0^-} \frac{F(x) - F(x-|h|)}{|h|}$$

By applying the same arguments over $[x-|h|, x]$, we can also conclude that $LF'(x) = f(x)$. Therefore, F is differentiable and

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = F'(x) = f(x) \quad \text{for all } x \in (a, b).$$

At $x = a$, by continuity and squeeze theorem, we also have

$$RF'(a) = \lim_{h \rightarrow 0^+} \frac{F(a+h) - F(a)}{h} = \lim_{h \rightarrow 0^+} \frac{\int_a^{a+h} f(t) dt}{h} = f(a)$$

Obviously, $LF'(b) = f(b)$ as well. Second: Suppose $g(x)$ is differentiable and $g'(x) = 0$ on $[a, b]$. By Lagrange's MVT, for any $x \in (a, b]$,

$$\frac{g(x) - g(a)}{x - a} = g'(c) = 0 \quad \text{for some } c \implies g(x) = g(a)$$

Therefore, $g(x)$ must be a constant function on $[a, b]$.

Now, by the first part of FTC,

$$\frac{d}{dx} \left(F(x) - \int_a^x F'(t) dt \right) = F'(x) - \frac{d}{dx} \int_a^x F'(t) dt = 0$$

for all $x \in [a, b]$. So,

$$g(x) = F(x) - \int_a^x F'(t) dt$$

must be a constant function on $[a, b]$. Hence,

$$F(b) - \int_a^b F'(t) dt = g(b) = g(a) = F(a)$$

and the result follows. $\square$

By the second part of FTC, for any continuous function $f(x)$ and $a, b \in \mathbb{R}$, we have:

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

Remark. Replacing $F(x)$ with $F(x) + C$ for any constant C would have no effect, as:

$$(F(b) + C) - (F(a) + C) = F(b) - F(a).$$

Example 6.32. Let us redo Example 6.29 by FTC:

$$\int_a^b f(x) dx = \int_a^b x^2 dx = \frac{1}{3}x^3 \Big|_a^b = \frac{1}{3}b^3 - \frac{1}{3}a^3.$$

Example 6.33. Consider the function $y = f(x) = x^3$ over the interval $[-1, 1]$.

Since $f(x) \geq 0$ when $x \in [0, 1]$ and $f(x) \leq 0$ when $x \in [-1, 0]$, to find the area of the region bounded by $y = f(x)$ and the x -axis, we need to split the interval $[-1, 1]$ into $[-1, 0]$ and $[0, 1]$ (to avoid cancellation):

$$\text{Area} = \left| \int_{-1}^0 x^3 dx \right| + \left| \int_0^1 x^3 dx \right| = \frac{1}{2}.$$

(In fact, the signed areas of the two regions will cancel out each other:

$$\int_{-1}^1 x^3 dx = 0.)$$

Example 6.34. When applying integration by substitution on definite integrals, there's no need to substitute x back in at the end as long as we adjust the bounds accordingly:

$$\begin{aligned}\int_0^1 (2x+1)^{\frac{3}{7}} dx &= \int_1^3 u^{\frac{3}{7}} \left(\frac{1}{2} du\right) \quad (\text{by letting } u = 2x+1) \\ &= \frac{1}{2} \cdot \frac{7}{10} u^{\frac{10}{7}} \Big|_1^3 = \frac{1}{2} \cdot \frac{7}{10} (3^{\frac{10}{7}} - 1).\end{aligned}$$

Example 6.35. •

$$\int_{-2}^{-1} \frac{1}{x} dx;$$

•

$$\int_{-3}^0 |x^2 + 3x + 2| dx;$$

•

$$\int_0^\pi x^6 \sin x dx.$$

Here's a surprising application of FTC.

Proposition 6.36. If $f(x)$ is a continuous function over $[0, 1]$, then:

$$\int_0^1 f(x) dx = \lim_{n \rightarrow +\infty} \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right)$$

Example 6.37.

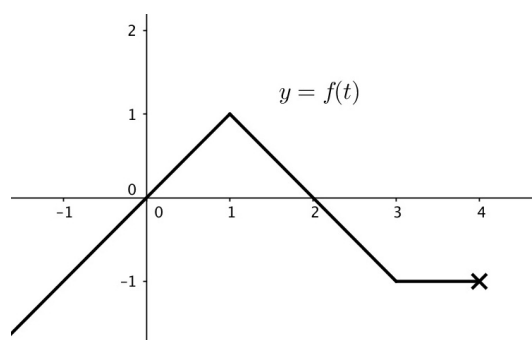
$$\begin{aligned}& \lim_{n \rightarrow +\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2} \right) \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n} \left(\frac{1}{1 + \left(\frac{1}{n}\right)^2} + \frac{1}{1 + \left(\frac{2}{n}\right)^2} + \cdots + \frac{1}{1 + \left(\frac{n}{n}\right)^2} \right) \\ &= \lim_{n \rightarrow +\infty} \sum_{k=1}^n \frac{1}{n} \cdot \frac{1}{1 + \left(\frac{k}{n}\right)^2} \\ &= \lim_{n \rightarrow +\infty} \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right)\end{aligned}$$

where $f(x) = \frac{1}{1+x^2}$.

Hence,

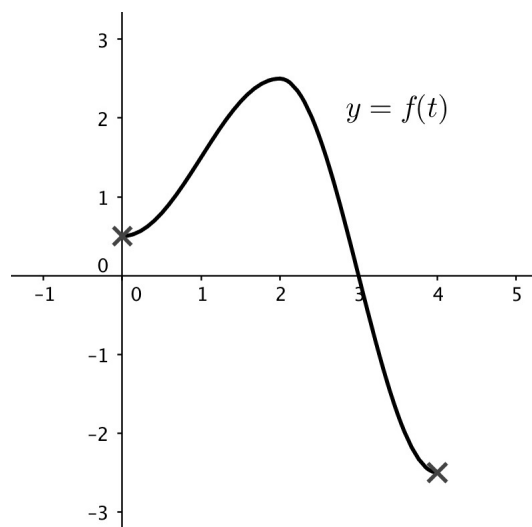
$$\lim_{n \rightarrow +\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2} \right) = \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}.$$

Example 6.38. Given that $F(x) = \int_2^x f(t) dt$ where



Find $F(2)$, $F(4)$ and $F(0)$.

Example 6.39. Given that $F(x) = \int_0^x f(t) dt$ where



Among $x \in [0, 4]$,

- When is $F(x)$ maximum?

- When is $F(x)$ minimum?
- When is $F'(x)$ maximum?

We may also generalize the first part of FTC a little bit:

Proposition 6.40. *If $f(x)$ is a continuous on $[a, b]$ and $h(x)$ is differentiable on $[c, d]$ such that $h([c, d]) \subseteq [a, b]$, then*

$$\frac{d}{dx} \int_a^{h(x)} f(t) dt = f(h(x)) h'(x) \quad \text{for all } x \in [c, d]$$

Proof of Proposition 6.40. Let $F(x) = \int_a^x f(t) dt$. Then, by FTC, $F(x)$ is differentiable over $[a, b]$. Moreover, for any $x \in [c, d]$,

$$\frac{d}{dx} \int_a^{h(x)} f(t) dt = \frac{d}{dx} F(h(x)) = F'(h(x)) h'(x) = f(h(x)) h'(x)$$

as desired. □

Example 6.41. Find $g'(x)$ if

•

$$g(x) = \int_{-1}^{x^2} \cos(t^2) dt;$$

•

$$g(x) = \int_{x^4}^1 \sec(\sqrt[3]{t}) dt;$$

•

$$g(x) = \int_{\sin x}^{\cos x} \ln(\sin t) dt.$$

6.13 Improper Integrals

Sometimes, we are interested in computing the definite integral of a function over $[a, +\infty)$ or $(-\infty, b]$.

Definition 6.42. If $f(x)$ is a continuous function over $[a, +\infty)$ such that the limit

$$\lim_{b \rightarrow +\infty} \int_a^b f(x) dx$$

exists, we define

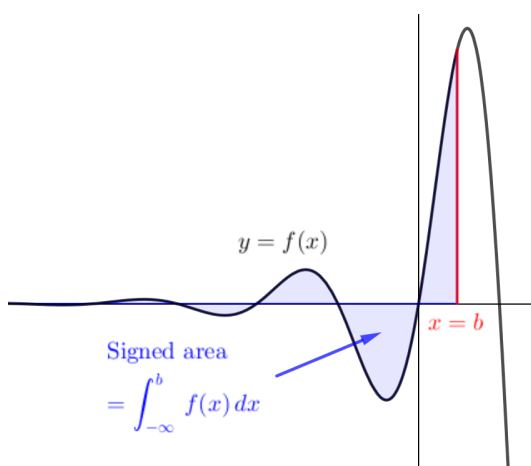
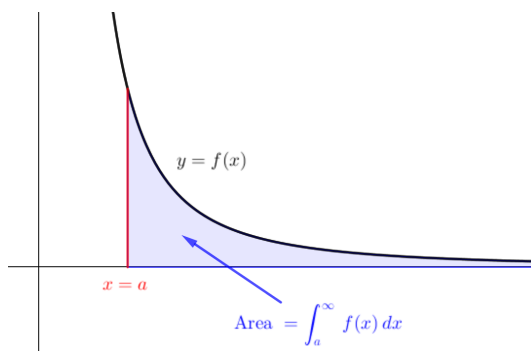
$$\int_a^{+\infty} f(x) dx = \lim_{b \rightarrow +\infty} \int_a^b f(x) dx$$

Similarly, if $f(x)$ is a continuous function over $(-\infty, b]$ such that the limit

$$\lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

exists, we define

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$



Occasionally, we may also consider one-sided **improper integrals**.

Definition 6.43. If $f(x)$ is a continuous function over $(a, b]$ such that the limit

$$\lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

exists, we define

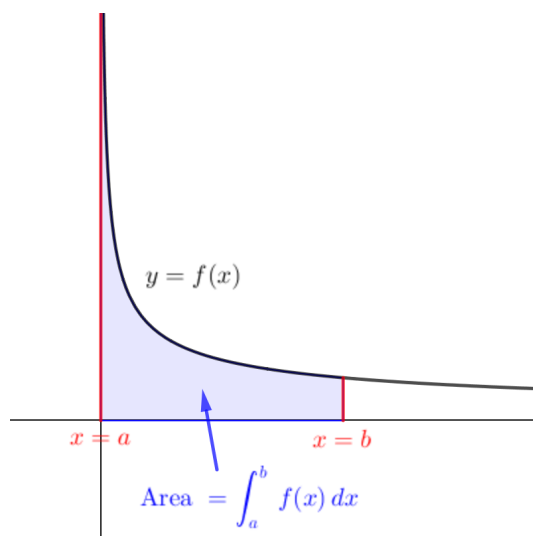
$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

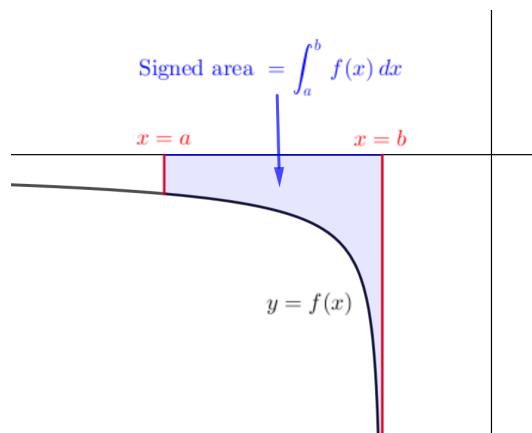
Similarly, if $f(x)$ is a continuous function over $[a, b)$ such that the limit

$$\lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

exists, we define

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$





In all four cases above, we say that the improper integral **converges** if the corresponding limit exists. Otherwise, we say that it **diverges**.

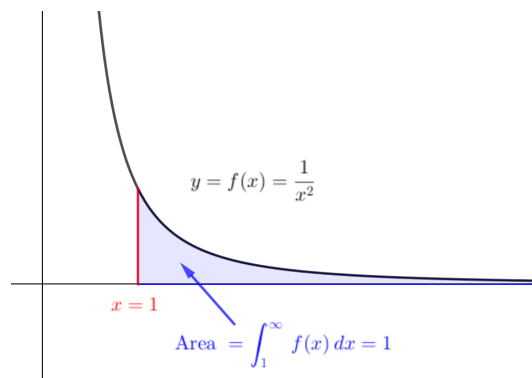
Example 6.44. Since

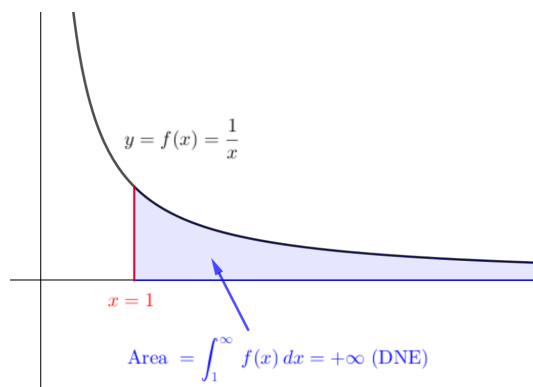
$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow +\infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \rightarrow +\infty} -\frac{1}{x} \Big|_1^b = 1,$$

we can conclude that $\int_1^{\infty} \frac{1}{x^2} dx$ converges to 1. On the other hand, since:

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow +\infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow +\infty} \ln |x| \Big|_1^b = +\infty \quad (\text{DNE}),$$

we can conclude that $\int_1^{\infty} \frac{1}{x} dx$ diverges to $+\infty$.





Example 6.45. Since:

$$\int_0^2 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0^+} \int_c^2 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0^+} 2\sqrt{x} \Big|_c^2 = 2\sqrt{2},$$

we can conclude that $\int_0^2 \frac{1}{\sqrt{x}} dx$ converges to $2\sqrt{2}$. On the other hand, since

$$\int_0^2 \frac{1}{x} dx = \lim_{c \rightarrow 0^+} \int_c^2 \frac{1}{x} dx = \lim_{c \rightarrow 0^+} \ln |x| \Big|_c^2 = +\infty \quad (\text{DNE}),$$

we can conclude that $\int_0^2 \frac{1}{x} dx$ diverges to $+\infty$.

