Ch 1 Fourier Series

Def: (1) <u>Trigonometric Series</u> (三角級數) on $[-\pi,\pi]$ is a series of functions of the form

$$\sum_{n=0}^{\infty}(a_n\cos nx + b_n\sin nx) \qquad (\text{where } a_n, b_n \in \mathbb{R})$$

$$= a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \qquad (b_0 = 0)$$

(2) If $b_n = 0$, $\forall n$, it is called a <u>cosine series</u>
If $a_n = 0$, $\forall n$, it is called a <u>sine series</u>

<u>Easy facts</u>

(1) If $\sum_{n=0}^{\infty}|a_n|$, $\sum_{n=0}^{\infty}|b_n| < \infty$

then $\sum_{n=0}^{\infty} a_n\cos nx + b_n\sin nx$

is <u>uniformly</u> and <u>absolutely</u> <u>convergent</u>

In particular, if $|a_n|, |b_n| \leq \frac{C}{n^s}$, $s>1$ (for some $C>0$)

then $\sum_{n=0}^{\infty}|a_n|$, $\sum_{n=0}^{\infty}|b_n| < \infty$ and hence

$\sum_{n=0}^{\infty} a_n\cos nx + b_n\sin nx$ is <u>uniformly</u> and <u>absolutely</u> <u>convergent</u>

(Pf: By M-test & $|\cos nx|, |\sin nx| \leq 1$)

(2) In this case,

$\phi(x) \overset{def}{=} \sum_{n=0}^{\infty} a_n\cos nx + b_n\sin nx$ is <u>continuous</u> on $[-\pi,\pi]$.

(3) $\phi(x)$ defined in (2) is 2π-periodic

Pf: $\phi(x+2\pi) = \lim_{n\to\infty} \sum_{k=0}^{n} \left[a_k \cos(k(x+2\pi)) + b_k \sin(k(x+2\pi))\right]$

$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \cos kx + b_k \sin kx$

$= \phi(x)$ #

Def: Let f be a 2π-periodic function on $\mathbb{R}$ which is Riemann integrable on $[-\pi,\pi]$. Then the Fourier Series (or Fourier expansion) of f is the trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with

$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ny\,dy \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\,dy \end{cases} \quad (n \geq 1)$$

Fourier Coefficients of f

Notes

(1) a_0 = average of f over $[-\pi,\pi]$

(2) Fourier series depends on the global information of f on $[-\pi,\pi]$.

(3) $f_1 \equiv f_2$ "almost everywhere" on $[-\pi,\pi]$
$\Rightarrow$ f_1 & f_2 have the same Fourier Series.

(4) Fourier series of f depends only on $f|_{(-\pi,\pi)}$, independent of the values of f on the end points.

($f_1 \equiv f_2$ "almost everywhere" means meas$(\{f_1 \neq f_2\}) = 0$,
i.e. $\forall \varepsilon > 0$, $\exists$ open intervals I_n, $n=1,2,\cdots$ s.t. $\{f_1 \neq f_2\} \subset \bigcup_{n=1}^{\infty} I_n$, $\sum_{n=1}^{\infty} |I_n| < \varepsilon$)

Motivation of the definition of Fourier Series

"If" $\quad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \forall x \in \mathbb{R}$

& "assume" uniformly convergent.

Then $\quad \int_{-\pi}^{\pi} f(x) \cos mx\,dx$

$$= a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx \right)$$

It is easy to calculate

$$\begin{cases} \bullet \ \int_{-\pi}^{\pi} \cos mx\,dx = \begin{cases} 2\pi & \text{if } m=0 \\ 0 & \text{if } m \neq 0 \end{cases} \\ \bullet \ \int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \begin{cases} \pi, & \text{if } m=n \\ 0, & \text{if } m \neq n \end{cases} \\ \bullet \ \int_{-\pi}^{\pi} \sin nx \cos mx\,dx = 0, \quad \forall\, n,m \geq 1 \end{cases} \quad \text{---} (*)_3$$

Hence if $m=0$,

$$\left.\begin{aligned} LHS &= \int_{-\pi}^{\pi} f(x)\,dx \\ RHS &= 2\pi a_0 \end{aligned}\right\} \Rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

If $m \neq 0$,

$$\left.\begin{aligned} LHS &= \int_{-\pi}^{\pi} f(x) \cos mx\,dx \\ RHS &= a_m \pi \end{aligned}\right\} \Rightarrow a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\,dx$$

Similarly, consider

$$\int_{-\pi}^{\pi} f(x) \sin mx\,dx = a_0 \int_{-\pi}^{\pi} \sin mx\,dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \sin mx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx\,dx \right)$$

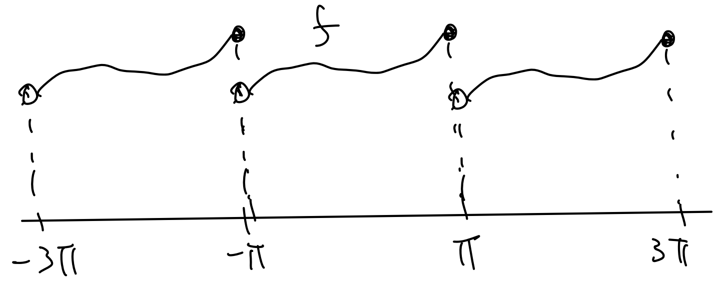
and using $\left\{\begin{array}{l} \bullet \int_{-\pi}^{\pi} \sin mx\,dx = 0 \quad \forall m \\ \bullet \int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases} \end{array}\right.$

(& using $(*)_3$)

$$\Rightarrow \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx, \quad \forall m \geq 1.$$

<u>Note</u>: For any Riemann integrable function f on $[-\pi,\pi]$, we can define all the a_0, a_n, b_n $(n\geq 1)$ as in the defn, and hence the <u>Fourier Series</u>.

On the other hand, we can restrict a f to $(-\pi,\pi]$ and extend <u>periodically</u> to a 2π-periodic function $\tilde{f}$ on $\mathbb{R}$



And according to the defn. of Fourier coefficients, f & $\tilde{f}$ have the <u>same</u> Fourier Series!

So we will <u>not</u> <u>distinguish</u> f & $\tilde{f}$.

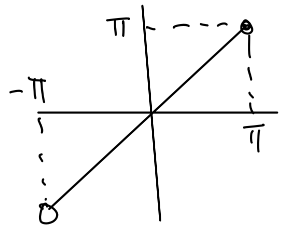
<u>Notation</u> We use $f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$

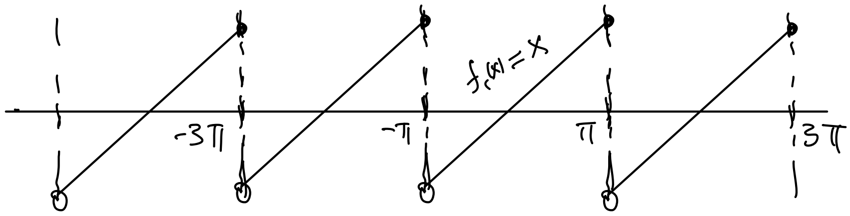
means "the trigonometric series on the RHS is the Fourier Series of f".

(does not indicate the series converges to f in any sense.)

eg 1.1 $f_1(x) = x$ restricted to $(-\pi, \pi]$

Extension to 2π-periodic function $\tilde{f}_1$ on $\mathbb{R}$





$$\begin{cases} a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} x\,dx = 0 \\[2ex] a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x\cos nx\,dx = 0 \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x\sin nx\,dx = (-1)^{n+1}\dfrac{2}{n} \end{cases} \quad \text{(check!)}$$

$$\therefore \quad f_1(x) = x \sim \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin nx$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$

(is a sine series)

($\because f_1$ is odd)