

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2023-2024 Term 1
Suggested Solutions of Homework Assignment 3
Due Date: 27 November 2023

1. Let $f(x) = \frac{x-1}{x-3}$.

- (a) Is Lagrange's mean value theorem applicable to f on the interval $[4, 5]$?
- (b) If your answer to part (a) is yes, find all possible values $c \in (4, 5)$, at which point(s) the tangent line to the graph is parallel to the secant line connecting the two end points $(4, f(4))$ and $(5, f(5))$.

Solution

- (a) Note that the function has a discontinuity at $x = 3$, while on the interval $[4, 5]$ it is continuous and differentiable, so the mean value theorem is applicable here. The derivative is

$$f'(x) = -\frac{2}{(x-3)^2}.$$

- (b) By the Lagrange's mean value theorem, let

$$f'(c) = \frac{f(5) - f(4)}{5 - 4} = \frac{2 - 3}{1} = -1,$$

choosing the root that lies in $(4, 5)$. Then $c = 3 + \sqrt{2}$.

2. By using Lagrange's mean value theorem, or otherwise, show that

- (a) $\sin x \leq x$ for all $x \in [0, +\infty)$.
- (b) $(1+x)^p \geq 1+px$ for any $p \geq 1$ and $x \geq 0$.

Solution

- (a) Let $f(x) = x - \sin x$. We want to show that $f(x) \geq 0$ for all $x \in [0, +\infty)$. Since $f(t)$ is continuous on $[0, x]$ and differentiable on $(0, x)$, one can apply Lagrange's MVT, then

$$1 - \cos c = f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Since $1 - \cos c \geq 0$ for any c , and $x \geq 0$, thus $f(x) \geq 0$ for all $x \in [0, +\infty)$.

- (b) The equality holds when $p = 1$ or $x = 0$. When $p > 1$, $x > 0$, let

$$f(x) = (1+x)^p - 1 - px.$$

It suffices to show that $f(x) > 0$ for all $x > 0$. For any $x > 0$, since f is continuous and differentiable on $[0, x]$, by Lagrange's MVT, there exists $c \in (0, x)$, such that

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = f'(c) = p(1+c)^{p-1} - p = p((1+c)^{p-1} - 1^{p-1}) > 0,$$

where the last inequality is due to that the function x^a for $a > 0$, $x > 0$ is a strictly increasing function. Therefore, $f(x) > 0$ for all $x \geq 0$.

3. Let $0 < a < b < \frac{\pi}{2}$. Prove that there exists $a < \xi < b$ such that

$$\ln \left(\frac{\cos a}{\cos b} \right) = (b - a) \tan \xi.$$

Solution

Fixed any $0 < a < b < \frac{\pi}{2}$, define $f : [a, b] \rightarrow \mathbb{R}$ by

$$f(x) = \ln \cos x$$

for any $x \in [a, b]$. Note that $\cos x$ is continuous on $[a, b]$, differentiable on (a, b) and

$$\cos b < \cos x < \cos a$$

for $0 < a < x < b < \frac{\pi}{2}$. Moreover, $\ln x$ is continuous on $[\cos b, \cos a]$ and differentiable on $(\cos b, \cos a)$.

Hence, f is continuous on $[a, b]$ and differentiable on (a, b) with

$$f'(x) = \frac{(\cos x)'}{\cos x} = -\tan x$$

for any $x \in (a, b)$.

Using Lagrange's mean value theorem, there exist some $\xi \in (a, b)$, such that

$$f'(\xi) = \frac{f(a) - f(b)}{a - b},$$

that is,

$$-\tan \xi = \frac{\ln \cos a - \ln \cos b}{a - b}.$$

Since $\ln x - \ln y = \ln\left(\frac{x}{y}\right)$, we have $\ln \cos a - \ln \cos b = \ln \left(\frac{\cos a}{\cos b} \right)$, and hence

$$\ln \left(\frac{\cos a}{\cos b} \right) = (b - a) \tan \xi.$$

4. Show that for all $0 < a < b \leq 1$,

$$(b-a)(1+\ln a) < \ln\left(\frac{b^b}{a^a}\right) < (b-a)(1+\ln b).$$

Solution

(a) Let $f(x) = x \ln x$ for $x > 0$. Consider $0 < a < b$, we know that f is continuous on $[a, b]$ and differentiable on (a, b) . By the Lagrange's mean value theorem, there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = \frac{b \ln b - a \ln a}{b - a} = f'(c) = 1 + \ln c.$$

Since $a < c < b$, $\ln a < \ln c < \ln b$. Thus

$$1 + \ln a < \frac{b \ln b - a \ln a}{b - a} < 1 + \ln b.$$

That is,

$$(b-a)(1+\ln a) < \ln\left(\frac{b^b}{a^a}\right) < (b-a)(1+\ln b).$$

5. Evaluate the following limits.

$$(a) \lim_{x \rightarrow 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$$

$$(d) \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{x}{x-1} \right)$$

$$(b) \lim_{x \rightarrow 0^+} \log_{\tan x} (\tan 2x)$$

$$(e) \lim_{x \rightarrow +\infty} \frac{e^{1+\ln x}}{\ln(1+e^x)}$$

$$(c) \lim_{x \rightarrow 0^+} \tan x \ln \sin x$$

Solution

(a) We compute the Taylor series of $\sin^{-1} x$ and $\tan^{-1} x$ at $x = 0$ to the third order:

$$(\sin^{-1})'(x) = (1-x^2)^{-\frac{1}{2}}$$

$$(\tan^{-1})'(x) = (1+x^2)^{-1}$$

$$(\sin^{-1})''(x) = x(1-x^2)^{-\frac{3}{2}}$$

$$(\tan^{-1})''(x) = -2x(1+x^2)^{-2}$$

$$(\sin^{-1})'''(x) = (1+2x^2)(1-x^2)^{-\frac{5}{2}}$$

$$(\tan^{-1})'''(x) = -2(1-x^2)(1+x^2)^{-3}$$

So the Taylor series are

$$\sin^{-1}(x) = x + \frac{x^3}{6} + O(x^4)$$

and

$$\tan^{-1}(x) = x - \frac{x^3}{3} + O(x^4)$$

Hence the limit is

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3} &= \lim_{x \rightarrow 0} \frac{(x + \frac{1}{6}x^3 + O(x^4)) - (x - \frac{1}{3}x^3 + O(x^4))}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^3 + O(x^4)}{x^3} \\
 &= \boxed{\frac{1}{2}}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \log_{\tan x}(\tan 2x) &= \lim_{x \rightarrow 0} \frac{\ln \tan 2x}{\ln \tan x} \\
 &= \lim_{x \rightarrow 0^+} \frac{(\ln \tan 2x)'}{(\ln \tan x)'} \\
 &= \lim_{x \rightarrow 0^+} 2 \frac{\tan x \cos^2 x}{\tan 2x \cos^2 2x} \\
 &= \lim_{x \rightarrow 0^+} 2 \frac{\sin 2x}{\sin 4x} \\
 &= \lim_{x \rightarrow 0^+} \frac{\sin 2x}{2x} \lim_{x \rightarrow 0^+} \frac{4x}{\sin 4x} \\
 &= \boxed{1}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \tan x \ln \sin x &= \lim_{x \rightarrow 0^+} \frac{\ln \sin x}{\tan x} \\
 &= \lim_{x \rightarrow 0^+} \frac{(\ln \sin x)'}{(\tan x)'} \\
 &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sin x} \cos x}{\frac{-1}{\tan^2 x} \sec^2 x} \\
 &= \lim_{x \rightarrow 0^+} -\sin x \cos x = \boxed{0}
 \end{aligned}$$

(d)

$$\begin{aligned}
 \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{x}{x-1} \right) &= \lim_{x \rightarrow 1} \frac{x-1-\ln x}{(x-1)\ln x} - 1 \\
 &= \lim_{x \rightarrow 1} \frac{(x-1-\ln x)'}{((x-1)\ln x)'} - 1 \\
 &= \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\ln x + \frac{x-1}{x}} - 1 \\
 &= \lim_{x \rightarrow 1} \frac{x-1}{x \ln x + x-1} - 1 \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)'}{(x \ln x + x-1)'} - 1 \\
 &= \lim_{x \rightarrow 1} \frac{1}{\ln x + 1} - 1 = \boxed{0}
 \end{aligned}$$

(e)

$$\begin{aligned}\lim_{x \rightarrow +\infty} \frac{e^{1+\ln x}}{\ln(1+e^x)} &= \lim_{x \rightarrow +\infty} \frac{xe}{\ln(1+e^x)} \\&= \lim_{x \rightarrow +\infty} \frac{(xe)'}{(\ln(1+e^x))'} \\&= \lim_{x \rightarrow +\infty} \frac{e}{\frac{1}{1+e^x} e^x} \\&= \lim_{x \rightarrow +\infty} e(1+e^{-x}) = \boxed{e}\end{aligned}$$

6. Evaluate the following limits.

(a) $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}}$

(c) $\lim_{x \rightarrow 0} \frac{(1+x)^x - 1}{x^2}$

(b) $\lim_{x \rightarrow 1} x^{\frac{2x}{x-1}}$

(d) $\lim_{x \rightarrow +\infty} \left(\frac{x^2 - 2x + 1}{x^2 - 4x + 2} \right)^x$

Solution

(a)

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1}{x^2} \ln \frac{\sin x}{x} &= \lim_{x \rightarrow 0} \frac{(\ln \frac{\sin x}{x})'}{(x^2)'} \\&= \lim_{x \rightarrow 0} \frac{\frac{x}{\sin x} \left(\frac{\cos x}{x} - \frac{\sin x}{x^2} \right)}{2x} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^2 \sin x} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{(x \cos x - \sin x)'}{(x^2 \sin x)'} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{-x \sin x}{2x \sin x + x^2 \cos x} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{-1}{2 + \frac{x}{\tan x}} \\&= \frac{1}{2} \frac{-1}{2+1} = -\frac{1}{6}\end{aligned}$$

So

$$\begin{aligned}\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}} &= e^{\lim_{x \rightarrow 0} \frac{1}{x^2} \ln \frac{\sin x}{x}} \\&= \boxed{e^{-\frac{1}{6}}}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{2x}{x-1} \ln x &= 2 \lim_{x \rightarrow 1} \frac{\ln x}{1 - \frac{1}{x}} \\&= 2 \lim_{x \rightarrow 1} \frac{(\ln x)'}{(1 - \frac{1}{x})'} \\&= 2 \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\frac{1}{x^2}} = 2\end{aligned}$$

So

$$\begin{aligned}\lim_{x \rightarrow 1} x^{\frac{2x}{x-1}} &= e^{\lim_{x \rightarrow 1} \frac{2x}{x-1} \ln x} \\&= \boxed{e^2}\end{aligned}$$

(c) We compute the Taylor series of $f(x) = (1+x)^x = e^{x \ln(1+x)}$ at $x = 0$ up to x^2 :

$$f'(x) = e^{x \ln(1+x)} (\ln(1+x) + 1 - \frac{1}{1+x})$$

$$f''(x) = e^{x \ln(1+x)} (\ln(1+x) + 1 - \frac{1}{1+x})^2 + e^{x \ln(1+x)} \frac{x+2}{(1+x)^2}$$

As $f(0) = e^{0 \ln 1} = 1$, $f'(0) = e^{0 \ln 1} (\ln 1 + 1 - \frac{0}{1+0}) = 0$, $f''(0) = e^{0 \ln 1} (\ln 1 + 1 - \frac{0}{1+0})^2 + e^{0 \ln 1} \frac{0+2}{(1+0)^2} = 2$, we have $(1+x)^x = 1 + x^2 + O(x^3)$, so

$$\lim_{x \rightarrow 0} \frac{(1+x)^x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{x^2 + O(x^3)}{x^2} = \boxed{1}$$

(d)

$$\begin{aligned}\lim_{x \rightarrow +\infty} x \ln \frac{(x-1)^2}{x^2 - 4x + 2} &= \lim_{x \rightarrow +\infty} \frac{\ln \frac{(x-1)^2}{x^2 - 4x + 2}}{x^{-1}} \\&= \lim_{x \rightarrow +\infty} \frac{(\ln \frac{(x-1)^2}{x^2 - 4x + 2})'}{(x^{-1})'} \\&= \lim_{x \rightarrow +\infty} \frac{2}{-x^{-2}} \frac{-x}{(x-1)(x^2 - 4x + 2)} = 2\end{aligned}$$

So

$$\lim_{x \rightarrow +\infty} \left(\frac{x^2 - 2x + 1}{x^2 - 4x + 2} \right)^x = e^{\lim_{x \rightarrow +\infty} x \ln \frac{(x-1)^2}{x^2 - 4x + 2}} = \boxed{e^2}$$

7. Find the x -intercepts, y -intercepts, asymptotes if there is any and sketch the graphs of the following functions.

(a) $y = \frac{x+5}{x-2}$

(b) $y = \frac{x^2 - 2}{x - 1}$

(c) $y = |4 + 3x - x^2|$

$$(d) \ y = x|x + 2|$$

$$(e) \ y = \left| \frac{7 - 2x}{x + 3} \right|$$

$$(f) \ y = \frac{1}{|x^2 - 4|}$$

Solution

(See next page for the graphs.)

$$(a) \text{ The } x\text{-intercept is at where } y = \frac{x+5}{x-2} = 0, \text{ so the } x\text{-intercept is } (-5, 0).$$

$$\text{The } y\text{-intercept is at where } x = 0, \text{ so the } y\text{-intercept is } (0, \frac{0+5}{0-2}) = (0, -\frac{5}{2}).$$

At $x = 2$, the denominator becomes 0, so $x = 2$ is a vertical asymptote.

Since $\lim_{x \rightarrow \pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x \rightarrow \pm\infty} y(x) = 1$, $y = 1$ is a horizontal asymptote.

$$(b) \text{ The } x\text{-intercept is at where } y = \frac{x^2-2}{x-1} = 0, \text{ so the } x\text{-intercepts are } (\sqrt{2}, 0) \text{ and } (-\sqrt{2}, 0).$$

$$\text{The } y\text{-intercept is at where } x = 0, \text{ so the } y\text{-intercept is } (0, \frac{0^2-2}{0-1}) = (0, 2).$$

At $x = 1$, the denominator becomes 0, so $x = 1$ is a vertical asymptote.

Since $\lim_{x \rightarrow \pm\infty} \frac{y(x)}{x} = 1$ and $\lim_{x \rightarrow \pm\infty} y(x) - x = 1$, $y = x + 1$ is an asymptote.

$$(c) \text{ The } x\text{-intercept is at where } y = |4 + 3x - x^2| = 0, \text{ so the } x\text{-intercepts are } (-1, 0) \text{ and } (4, 0).$$

$$\text{The } y\text{-intercept is at where } x = 0, \text{ so the } y\text{-intercept is } (0, |4 + 3 \cdot 0 - 0^2|) = (0, 4).$$

Since the function has no singularity and $\lim_{x \rightarrow \pm\infty} \frac{y(x)}{x} = \pm\infty$, the function has no asymptote.

$$(d) \text{ The } x\text{-intercept is at where } y = x|x + 2| = 0, \text{ so the } x\text{-intercepts are } (0, 0) \text{ and } (-2, 0).$$

$$\text{The } y\text{-intercept is at where } x = 0, \text{ so the } y\text{-intercept is } (0, 0 \cdot |0 + 2|) = (0, 0).$$

Since the function has no singularity and $\lim_{x \rightarrow \pm\infty} \frac{y(x)}{x} = +\infty$, the function has no asymptote.

$$(e) \text{ The } x\text{-intercept is at where } y = \left| \frac{7-2x}{x+3} \right| = 0, \text{ so the } x\text{-intercept is } (\frac{7}{2}, 0).$$

$$\text{The } y\text{-intercept is at where } x = 0, \text{ so the } y\text{-intercept is } (0, \left| \frac{7-2 \cdot 0}{0+3} \right|) = (0, \frac{7}{3}).$$

At $x = -3$, the denominator becomes 0, so $x = -3$ is a vertical asymptote.

Since $\lim_{x \rightarrow \pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x \rightarrow \pm\infty} y(x) = 2$, $y = 2$ is an asymptote.

$$(f) \text{ The } x\text{-intercept is at where } y = \frac{1}{|x^2-4|} = 0, \text{ so the function has no } x\text{-intercept.}$$

$$\text{The } y\text{-intercept is at where } x = 0, \text{ so the } y\text{-intercept is } (0, \frac{1}{|0^2-4|}) = (0, \frac{1}{4}).$$

At $x = -2$ and at $x = 2$, the denominator becomes 0, so $x = 2$ and $x = -2$ are vertical asymptotes.

Since $\lim_{x \rightarrow \pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x \rightarrow \pm\infty} y(x) = 0$, $y = 0$ is an asymptote.

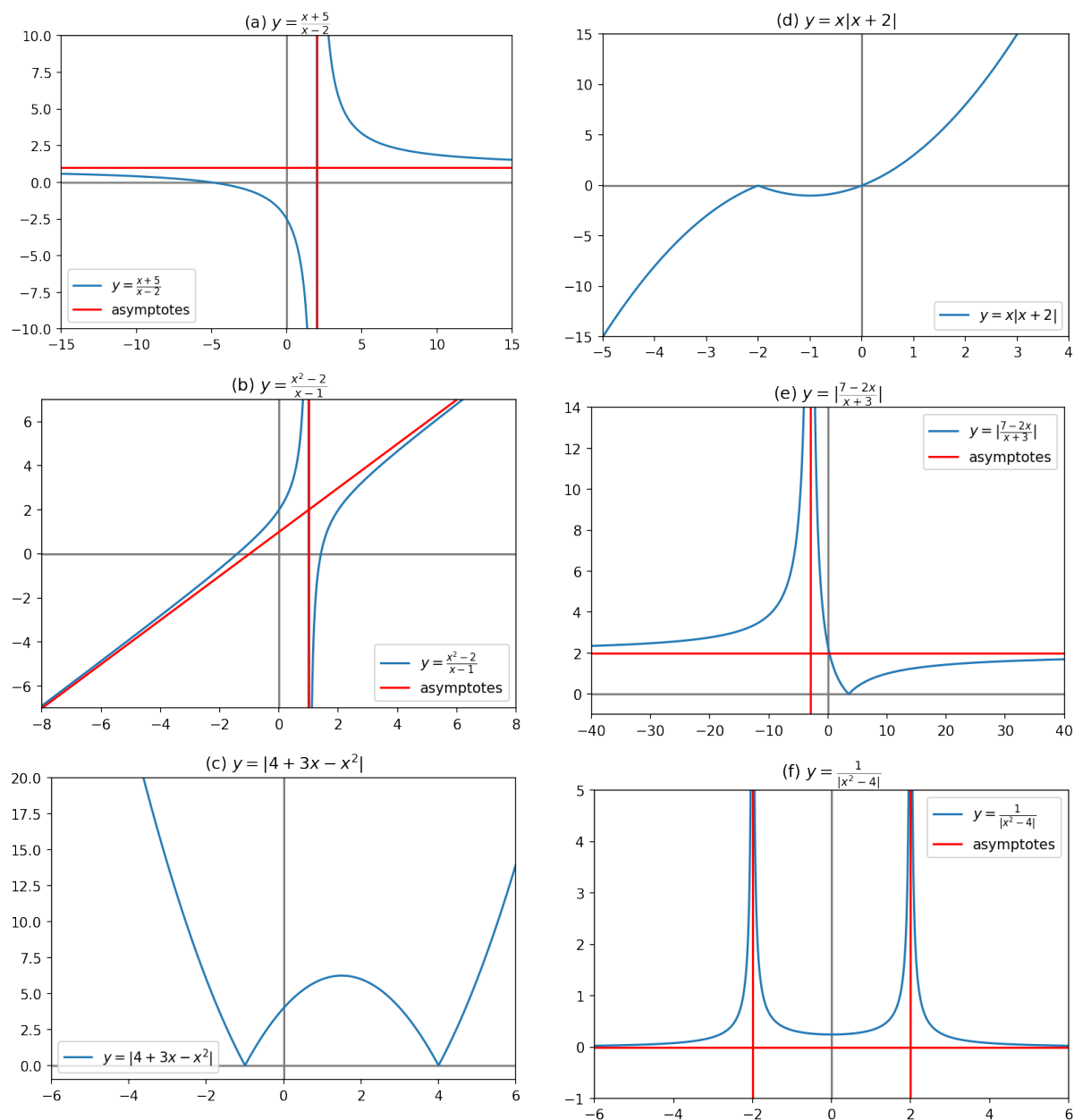


Figure 1: The graphs of the functions for question 7. Asymptotes, if they exist, are also drawn.

8. For each of the following functions $f(x)$, find

- $f'(x)$ and $f''(x)$.
- range of values of x for which $f(x)$ is increasing.
- asymptotes of $y = f(x)$.
- all relative extremum points

Then sketch the graph of $y = f(x)$.

$$(a) \quad f(x) = \frac{x}{(x-2)^2}$$

$$(c) \quad f(x) = \frac{x^2}{x^2 - 2x + 2}$$

$$(b) \quad f(x) = \frac{x^2 + 5x + 7}{x + 2}$$

$$(d) \quad f(x) = x^{\frac{2}{3}} - 1$$

Solution

(See next page for the graphs.)

(a)

$$f'(x) = \frac{d}{dx} \frac{x}{(x-2)^2} = \frac{1}{(x-2)^2} - \frac{2x}{(x-2)^3} = \boxed{-\frac{x+2}{(x-2)^3}}$$

$$f''(x) = \frac{d}{dx} -\frac{x+2}{(x-2)^3} = -\left(\frac{1}{(x-2)^3} - \frac{3(x+2)}{(x-2)^4}\right) = \boxed{\frac{2x+8}{(x-2)^4}}$$

f is differentiable on the domain $(-\infty, 2) \cup (2, \infty)$, and $f'(x) > 0$ if and only if $-2 < x < 2$. So f is increasing on $\boxed{[-2, 2]}$.

Since when $x = 2$, the denominator becomes 0, so $\boxed{x = 2}$ is a vertical asymptote.

As $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = 0$ and $\lim_{x \rightarrow \pm\infty} f(x) = 0$, $\boxed{y = 0}$ is an asymptote.

The only critical point of $f(x)$ is $x = -2$, at which $f''(-2) = \frac{1}{64} > 0$, so $\boxed{x = -2}$ is the only relative extremum and is a relative minimum.

(b)

$$f'(x) = \frac{d}{dx} \frac{x^2 + 5x + 7}{x + 2} = \frac{2x + 5}{x + 2} - \frac{x^2 + 5x + 7}{(x + 2)^2} = \frac{x^2 + 4x + 3}{(x + 2)^2} = \boxed{\frac{(x+1)(x+3)}{(x+2)^2}}$$

$$f''(x) = \frac{d}{dx} \frac{x^2 + 4x + 3}{(x + 2)^2} = \frac{2x + 4}{(x + 2)^2} - (x^2 + 4x + 3) \frac{-2}{(x + 2)^3} = \boxed{\frac{2}{(x + 2)^3}}$$

f is differentiable on the domain $(-\infty, -2) \cup (-2, \infty)$, and $f'(x) > 0$ if and only if $x < -3$ or $-1 < x$. Also, $f(-3) = -1 < 3 = f(-1)$. So f is increasing on $\boxed{(-\infty, -3] \cup [-1, \infty)}$.

Since when $x = -2$, the denominator becomes 0, so $\boxed{x = -2}$ is a vertical asymptote.

As $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = 1$ and $\lim_{x \rightarrow \pm\infty} f(x) - x = 3$, so $\boxed{y = x + 3}$ is an asymptote.

The only critical points are $x = -1$ and $x = -3$. Since $f''(-1) = 2 > 0$ and $f''(-3) = -2 < 0$, so the only relative extrema are at $x = -1$ and $x = -3$, where $\boxed{x = -1}$ is a relative minimum and $\boxed{x = -3}$ is a relative maximum.

(c)

$$f'(x) = \frac{d}{dx} \frac{x^2}{x^2 - 2x + 2} = \frac{2x}{x^2 - 2x + 2} - \frac{x^2(2x - 2)}{(x^2 - 2x + 2)^2} = \boxed{-\frac{2x(x-2)}{(x^2 - 2x + 2)^2}}$$

$$f''(x) = \frac{-4x + 4}{(x^2 - 2x + 2)^2} - \frac{2(-2x^2 + 4x)(2x - 2)}{(x^2 - 2x + 2)^3} = \boxed{\frac{4(x-1)(x^2 - 2x - 2)}{(x^2 - 2x + 2)^3}}$$

f is differential on the domain $(-\infty, \infty)$, and $f'(x) > 0$ if and only if $0 < x < 2$, so f is increasing on $[0, 2]$.

As $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = 0$ and $\lim_{x \rightarrow \pm\infty} f(x) = 1$, $y = 1$ is an asymptote.

The critical points of f are $x = 0$ and $x = 2$. Since $f''(0) = 1 > 0$ and $f''(2) = -1 < 0$, so the only relative extrema are $x = 0$ and $x = 2$, where $x = 0$ is a relative minimum and $x = 2$ is a relative maximum.

(d)

$$f'(x) = \frac{d}{dx}(x^{\frac{2}{3}} - 1) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$f''(x) = \frac{d}{dx} \frac{2}{3}x^{-\frac{1}{3}} = -\frac{2}{9}x^{-\frac{4}{3}} = -\frac{2}{9\sqrt[3]{x^4}}$$

f is differentiable on $(-\infty, 0) \cup (0, \infty)$, and $f'(x) > 0$ if and only if $x > 0$. So f is increasing on $[0, \infty)$.

Since $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = 0$ but $\lim_{x \rightarrow \pm\infty} f(x)$ does not exist. So f has no asymptote.

The only critical points of f are $x = 0$ as f is not differentiable at $x = 0$ and $f'(x) \neq 0$ on $(-\infty, 0) \cup (0, \infty)$. Since for $x \neq 0$, $f(x) = -1 + \sqrt[3]{x^2} \geq -1 = f(0)$, $x = 0$ is the only relative extremum and is a relative minimum.

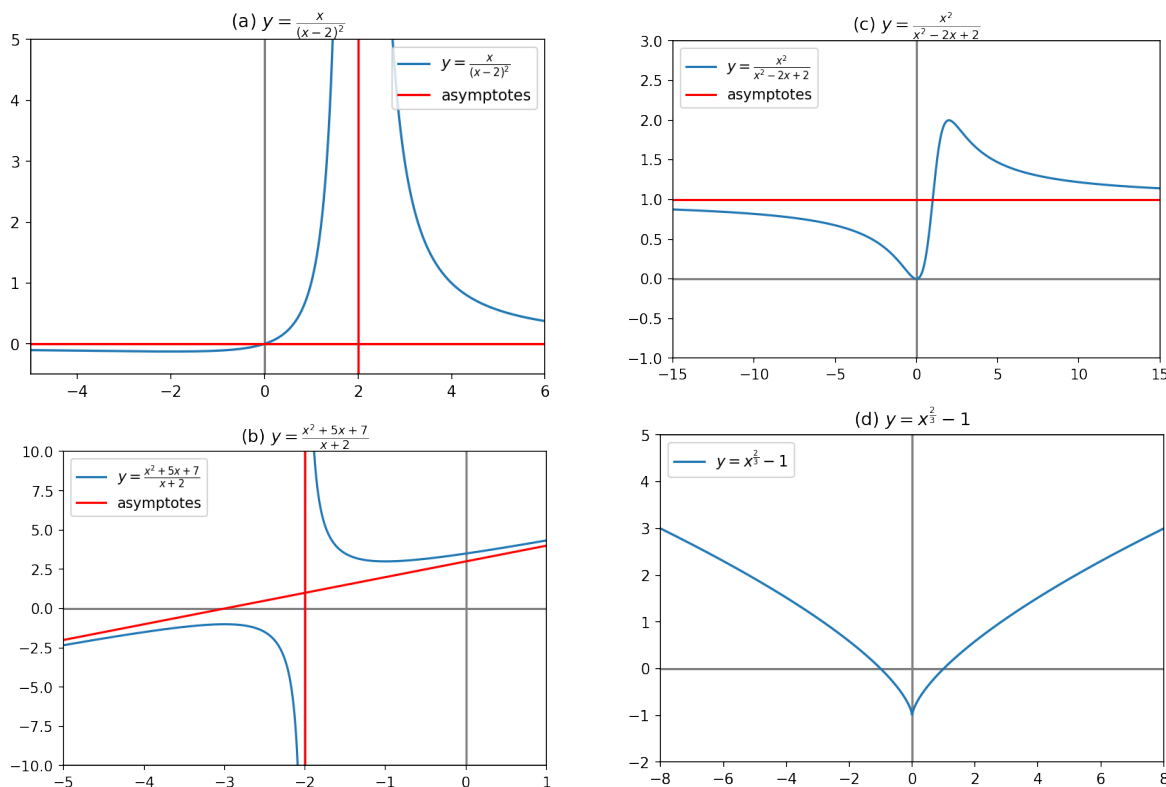


Figure 2: The graphs for question 8. Asymptotes, if they exist, are also drawn.

9. For each of the following functions $f(x)$, find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ and the Taylor series up to the term in x^3 of $f(x)$ about the point $x = 0$.

(a) $f(x) = \ln \cos x$

(b) $f(x) = e^x \sin x$

Solution

(a) $f(0) = \ln \cos 0 = 0$

$$f'(x) = \frac{d}{dx} \ln \cos x = \frac{1}{\cos x} (-\sin x) = -\tan x$$

So $f'(0) = -\tan 0 = 0$

$$f''(x) = \frac{d}{dx} -\tan x = -\sec^2 x$$

So $f''(0) = -\sec^2 0 = -1$

$$f'''(x) = \frac{d}{dx} -\sec^2 x = \frac{2}{\cos^3} (-\sin x) = -2 \tan x \sec^2 x$$

So $f'''(0) = 0$

So the Taylor series of $f(x) = \ln \cos x$ about $x = 0$ up to x^3 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + O(x^4) = \boxed{-\frac{1}{2}x^2 + O(x^4)}$$

(b) $f(0) = e^0 \sin 0 = 0$

$$f'(x) = \frac{d}{dx} e^x \sin x = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)$$

So $f'(0) = e^0 (\sin 0 + \cos 0) = 1$

$$f''(x) = \frac{d}{dx} e^x (\sin x + \cos x) = e^x (\sin x + \cos x) + e^x (\cos x - \sin x) = 2e^x \cos x$$

So $f''(0) = 2e^0 \cos 0 = 2$

$$f'''(x) = \frac{d}{dx} 2e^x \cos x = 2(e^x \cos x - e^x \sin x) = 2e^x (\cos x - \sin x)$$

So $f'''(0) = 2e^0 (\cos 0 - \sin 0) = 2$

So the Taylor series of $f(x) = e^x \sin x$ about $x = 0$ up to x^3 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + O(x^4) = \boxed{x + x^2 + \frac{1}{3}x^3 + O(x^4)}$$

10. Find the Taylor series up to the term in $(x - c)^3$ of the functions about $x = c$.

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|------------------------------------|-------------------------------------|
| (a) $\frac{1}{1+x}; c = 1.$ | (e) $\sin^2 x; c = 0$ |
| (b) $\frac{2-x}{3+x}; c = 1.$ | (f) $\ln x; c = e.$ |
| (c) $\frac{x}{(x-1)(x-2)}; c = 0.$ | (g) $3^x; c = 0.$ |
| (d) $\cos x; c = \frac{\pi}{4}.$ | (h) $\sqrt{2+x}; c = 1.$ |
| | (i) $\frac{1}{\sqrt{7-3x}}; c = 1.$ |

Solution

(a) Let $f(x) = \frac{1}{1+x}$. Then $f(c) = \frac{1}{1+c} = \frac{1}{2}$, $f'(c) = \frac{-1}{(1+c)^2} = -\frac{1}{4}$, $f''(c) = \frac{2}{(1+c)^3} = \frac{1}{4}$, $f'''(c) = \frac{-6}{(1+c)^4} = -\frac{3}{8}$.

$$\text{So } \frac{1}{1+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$$

$$= \boxed{\frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + O((x-1)^4)}$$

(b) Let $f(x) = \frac{2-x}{3+x} = -1 + \frac{5}{3+x}$. Then $f(c) = -1 + \frac{5}{3+c} = \frac{1}{4}$, $f'(c) = \frac{-5}{(3+c)^2} = -\frac{5}{16}$, $f''(c) = \frac{10}{(3+c)^3} = \frac{5}{32}$, $f'''(c) = \frac{-30}{(3+c)^4} = -\frac{15}{128}$.

$$\text{So } \frac{2-x}{3+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$$

$$= \boxed{\frac{1}{4} - \frac{5}{16}(x-1) + \frac{5}{64}(x-1)^2 - \frac{5}{256}(x-1)^3 + O((x-1)^4)}$$

(c) Let $f(x) = \frac{x}{(x-1)(x-2)}$. Then $f(c) = \frac{0}{(0-1)(0-2)} = 0$, $f'(c) = -\frac{c^2-2}{(c-1)^2(c-2)^2} = \frac{1}{2}$, $f''(c) = \frac{2(c^3-6c+6)}{(c-1)^3(c-2)^3} = \frac{3}{2}$, $f'''(c) = -\frac{6(c^4-12c^2+24c-14)}{(c-1)^4(c-2)^4} = \frac{21}{4}$.

$$\text{So } \frac{x}{(x-1)(x-2)} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$$

$$= \boxed{\frac{1}{2}x + \frac{3}{4}x^2 + \frac{7}{8}x^3 + O(x^4)}$$

(d) Let $f(x) = \cos x$. Then $f(c) = \cos c = \frac{\sqrt{2}}{2}$, $f'(c) = -\sin c = -\frac{\sqrt{2}}{2}$, $f''(c) = -\cos c = -\frac{\sqrt{2}}{2}$, $f'''(c) = \sin c = \frac{\sqrt{2}}{2}$.

$$\text{So } \cos x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$$

$$= \boxed{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x - \frac{\pi}{4})^2 + \frac{\sqrt{2}}{12}(x - \frac{\pi}{4})^3 + O((x - \frac{\pi}{4})^4)}$$

(e) Let $f(x) = \sin^2 x$. Then $f(c) = \sin^2 c = 0$, $f'(c) = \sin(2c) = 0$, $f''(c) = 2\cos(2c) = 2$, $f'''(c) = -4\sin(2c) = 0$.

$$\text{So } \sin^2 x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$$

$$= \boxed{x^2 + O(x^4)}$$

(f) Let $f(x) = \ln x$. Then $f(c) = \ln c = 1$, $f'(c) = \frac{1}{c} = \frac{1}{e}$, $f''(c) = -\frac{1}{c^2} = -\frac{1}{e^2}$, $f'''(c) = \frac{2}{c^3} = \frac{2}{e^3}$.

$$\text{So } \ln x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$$

$$= \boxed{1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + O((x-e)^4)}$$

(g) Let $f(x) = 3^x$. Then $f(c) = 3^c = 1$, $f'(c) = 3^c \ln 3 = \ln 3$, $f''(c) = 3^c (\ln 3)^2 = (\ln 3)^2$, $f'''(c) = 3^c (\ln 3)^3 = (\ln 3)^3$.

So $3^x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$$= \boxed{1 + x \ln 3 + \frac{(\ln 3)^2}{2}x^2 + \frac{(\ln 3)^3}{6}x^3 + O(x^4)}$$

(h) Let $f(x) = \sqrt{2+x}$. Then $f(c) = \sqrt{2+c} = \sqrt{3}$, $f'(c) = \frac{1}{2}(2+c)^{-\frac{1}{2}} = \frac{\sqrt{3}}{6}$, $f''(c) = -\frac{1}{4}(2+c)^{-\frac{3}{2}} = -\frac{\sqrt{3}}{36}$, $f'''(c) = \frac{3}{8}(2+c)^{-\frac{5}{2}} = \frac{\sqrt{3}}{72}$.

So $\sqrt{2+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$$= \boxed{\sqrt{3} + \frac{\sqrt{3}}{6}(x-1) - \frac{\sqrt{3}}{72}(x-1)^2 + \frac{\sqrt{3}}{432}(x-1)^3 + O((x-1)^4)}$$

(i) Let $f(x) = \frac{1}{\sqrt{7-3x}}$. Then $f(c) = \frac{1}{\sqrt{7-3c}} = \frac{1}{2}$, $f'(c) = -\frac{1}{2}(7-3c)^{-\frac{3}{2}} = -\frac{3}{16}$, $f''(c) = \frac{27}{3}(7-3c)^{-\frac{5}{2}} = \frac{27}{128}$, $f'''(c) = \frac{405}{8}(7-3c)^{-\frac{7}{2}} = \frac{405}{1024}$.

So $\frac{1}{\sqrt{7-3x}} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$$= \boxed{\frac{1}{2} + \frac{3}{16}(x-1) + \frac{27}{256}(x-1)^2 + \frac{135}{2048}(x-1)^3 + O((x-1)^4)}$$

Alternatively, by using the Taylor series of the elementary functions,

$$(a) \frac{1}{x+1} = \frac{1}{2} \frac{1}{1+\frac{x-1}{2}} = \frac{1}{2} \left(1 - \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^3 + O((x-1)^4) \right)$$

$$= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + O((x-1)^4)$$

$$(b) \frac{2-x}{3+x} = -1 + \frac{5}{4} \frac{1}{1+\frac{x-1}{4}} = -1 + \frac{5}{4} \left(1 - \frac{x-1}{4} + \left(\frac{x-1}{4}\right)^2 - \left(\frac{x-1}{4}\right)^3 + O((x-1)^4) \right)$$

$$= \frac{1}{4} - \frac{5}{16}(x-1) + \frac{5}{64}(x-1)^2 - \frac{5}{256}(x-1)^3 + O((x-1)^4)$$

$$(c) \frac{x}{(x-1)(x-2)} = -\frac{1}{1-\frac{x}{2}} + \frac{1}{1-x}$$

$$= -\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + O(x^4)\right) + (1 + x + x^2 + x^3 + O(x^4))$$

$$= \frac{1}{2}x + \frac{3}{4}x^2 + \frac{7}{8}x^3 + O(x^4)$$

$$(d) \cos x = \cos\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \left(\cos\left(x - \frac{\pi}{4}\right) - \sin\left(x - \frac{\pi}{4}\right) \right)$$

$$= \frac{\sqrt{2}}{2} \left(\left(1 - \frac{(x-\frac{\pi}{4})^2}{2} + O((x-\frac{\pi}{4})^4)\right) - \left((x-\frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^3}{6} + O((x-\frac{\pi}{4})^4)\right) \right)$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 + \frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3 + O((x-\frac{\pi}{4})^4)$$

$$(e) \sin^2 x = \frac{1}{2}(1 - \cos(2x)) = \frac{1}{2}(1 - (1 - \frac{(2x)^2}{2} + O(x^4)))$$

$$= x^2 + O(x^4)$$

$$(f) \ln x = 1 + \ln\left(1 + \frac{x-e}{e}\right) = 1 + \left(\frac{x-e}{e} - \frac{1}{2}\left(\frac{x-e}{e}\right)^2 + \frac{1}{3}\left(\frac{x-e}{e}\right)^3 + O((x-e)^4)\right)$$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + O((x-e)^4)$$

$$(g) 3^x = e^{x \ln 3} = 1 + x \ln 3 + \frac{1}{2}(x \ln 3)^2 + \frac{1}{6}(x \ln 3)^3 + O(x^4)$$

$$= 1 + x \ln 3 + \frac{(\ln 3)^2}{2}x^2 + \frac{(\ln 3)^3}{6}x^3 + O(x^4)$$

$$(h) \sqrt{2+x} = \sqrt{3}\left(1 + \frac{x-1}{3}\right)^{\frac{1}{2}}$$

$$= \sqrt{3} \left(1 + \frac{1}{2} \frac{x-1}{3} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} \left(\frac{x-1}{3}\right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{6} \left(\frac{x-1}{3}\right)^3 + O((x-1)^4) \right)$$

$$= \sqrt{3} + \frac{\sqrt{3}}{6}(x-1) - \frac{\sqrt{3}}{72}(x-1)^2 + \frac{\sqrt{3}}{432}(x-1)^3 + O((x-1)^4)$$

$$\begin{aligned}
\text{(i)} \quad \frac{1}{\sqrt{7-3x}} &= \frac{1}{2} \left(1 - \frac{x-1}{4/3}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2} \left(1 - \frac{-1}{2} \frac{x-1}{4/3} + \frac{-1}{2} \left(\frac{-1}{2} - 1\right) \frac{(x-1)^2}{(4/3)^2} - \frac{-1}{2} \left(\frac{-1}{2} - 1\right) \left(\frac{-1}{2} - 2\right) \frac{(x-1)^3}{(4/3)^3} + O((x-1)^4)\right) \\
&= \frac{1}{2} + \frac{3}{16}(x-1) + \frac{27}{256}(x-1)^2 + \frac{135}{2048}(x-1)^3 + O((x-1)^4)
\end{aligned}$$

11. (a) Find $\frac{d^2y}{dx^2}$ at $(1, 0)$, if

$$y^3 + y = x^3 - x.$$

- (b) Find the Taylor polynomial of order 3 around $x = 0$ for $f(x) = e^{\cos x}$.

Solution

- (a) By implicit differentiation,

$$3y^2y' + y' = 3x^2 - 1.$$

Let $x = 1$, $y = 0$, solve for $y' = 2$. Now that

$$3y^2y'' + 6y(y')^2 + y'' = 6x,$$

plug in $x = 1$, $y = 0$, then $y'' = 6$ at the point $(1, 0)$.

- (b) By Taylor series, since $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$, and $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$, thus

$$\begin{aligned}
e^{\cos x} &= 1 + \left(1 - \frac{x^2}{2}\right) + \frac{1}{2} \left(1 - \frac{x^2}{2}\right)^2 + \frac{1}{3!} \left(1 - \frac{x^2}{2}\right)^3 + \dots \\
&= 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots - \frac{1}{2} \left(x^2 + x^2 + \frac{x^2}{2} + \frac{x^2}{3!} + \frac{x^2}{4!} + \dots\right) \\
&= \boxed{e - \frac{e}{2}x^2}.
\end{aligned}$$

12. By considering appropriate Taylor series expansions, evaluate the limits below:

$$\begin{aligned}
\text{(a)} \quad \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{\ln(1 + 2x)} & \qquad \text{(c)} \quad \lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{1 - \sqrt{1 - x^3}} \\
\text{(b)} \quad \lim_{x \rightarrow 0} \left(\frac{1}{\ln(1 + 2x)} + \frac{1}{\ln(1 - 2x)} \right) & \qquad \text{(d)} \quad \lim_{x \rightarrow 0} \frac{e^{3x} - \sin x - \cos x + \ln(1 - 2x)}{-1 + \cos(5x)}
\end{aligned}$$

Solution

- (a) Note $e^{2x} = 1 + 3x + \frac{(3x)^2}{2} + \dots$
and $\ln(1 + 2x) = 2x - \frac{1}{2}(2x)^2 + \dots$

Then

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{\ln(1 + 2x)} &= \lim_{x \rightarrow 0} \frac{1 + 3x + \frac{9}{2}x^2 + \dots - 1}{2x - 2x^2 + \dots} \\
&= \lim_{x \rightarrow 0} \frac{3x + \frac{9}{2}x^2 + \dots}{2x - 2x^2 + \dots} = \lim_{x \rightarrow 0} \frac{3 + \frac{9}{2}x + \dots}{2 - 2x + \dots} \\
&= \boxed{\frac{3}{2}}.
\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x \rightarrow 0} \left(\frac{1}{\ln(1+2x)} + \frac{1}{\ln(1-2x)} \right) &= \lim_{x \rightarrow 0} \frac{\ln(1-2x) + \ln(1+2x)}{\ln(1+2x) \ln(1-2x)} \\ &= \lim_{x \rightarrow 0} \frac{\ln(1-4x^2)}{\ln(1+2x) \ln(1-2x)} \\ &= \lim_{x \rightarrow 0} \frac{-4x^2 - \frac{(2x)^4}{2} + \dots}{\left(x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3}\right) \left(-2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} + \dots\right)} = \lim_{x \rightarrow 0} \frac{-4x^2 + o(x^2)}{-2x^2 + o(x^2)} = \boxed{2}.\end{aligned}$$

(c) Note

$$\begin{aligned}1 - \cos x &= 1 - \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \dots\right) \\ &= \frac{1}{2}x^2 - \frac{1}{4!}x^4 + \dots\end{aligned}$$

So

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{1 - \sqrt{1 - x^3}} &= \lim_{x \rightarrow 0} \frac{x(1 - \cos x)(1 + \sqrt{1 - x^3})}{1 - (1 - x^3)} \\ &= \lim_{x \rightarrow 0} \frac{x(1 - \cos x)(1 + \sqrt{1 - x^3})}{x^3}\end{aligned}$$

Note

$$\begin{aligned}\lim_{x \rightarrow 0} (1 + \sqrt{1 - x^3}) &= 2, \\ \text{and } \lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^3 - \frac{1}{4!}x^5 + \dots}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} - \frac{1}{4!}x^2 + \dots}{1} = \frac{1}{2}.\end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{1 - \sqrt{1 - x^3}} = 2 \cdot \frac{1}{2} = \boxed{1}$$

(d) Note that

$$\begin{aligned}e^{3x} &= 1 + 3x + \frac{9}{2}x^2 + o(x^2), \\ \sin x &= x + o(x^2), \\ \cos x &= 1 - \frac{1}{2}x^2 + o(x^2), \\ \ln(1 - 2x) &= -2x - 2x^2 + o(x^2), \\ \cos(5x) &= 1 - \frac{25}{2}x^2 + o(x^2).\end{aligned}$$

Thus

$$\begin{aligned}& \lim_{x \rightarrow 0} \frac{e^{3x} - \sin x - \cos x + \ln(1 - 2x)}{-1 + \cos(5x)} \\&= \lim_{x \rightarrow 0} \frac{1 + 3x + \frac{9}{2}x^2 - x - x + \frac{x^2}{2} - 2x - 2x^2 + o(x^2)}{-1 + 1 - \frac{25}{2}x^2 + o(x^2)} \\&= \boxed{-\frac{6}{25}}.\end{aligned}$$