THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2023-2024 Term 1
Homework Assignment 2 (Solution)
Due Date: October 30, 2023 (Monday)

1. The function f is continuous at $x = 0$ and is defined for $-1 < x < 1$ by

$$f(x) = \begin{cases} \frac{2a}{x}(e^x - 1) & \text{if } -1 < x < 0 \\ 1 & \text{if } x = 0 \\ \frac{bx\cos x}{1-\sqrt{1-x}} & \text{if } 0 < x < 1. \end{cases}$$

Determine the values of the constants a and b.

Solution For f to be continuous at $x = 0$,

(a) $\lim_{x\to 0+} f(x) = f(0)$

$$\begin{aligned} 1 &= \lim_{x\to 0+} \frac{bx\cos x}{1-\sqrt{1-x}} \\ &= \lim_{x\to 0+} \frac{bx\cos x(1+\sqrt{1-x})}{1-(1-x)} \\ &= \lim_{x\to 0+} b\cos x(1+\sqrt{1-x}) \\ &= 2b \end{aligned}$$

So $b = \frac{1}{2}$.

(b) $\lim_{x\to 0-} f(x) = f(0)$

$$\begin{aligned} 1 &= \lim_{x\to 0-} \frac{2a}{x}(e^x - 1) \\ &= 2a \end{aligned}$$

So $a = \frac{1}{2}$.

2. Determine whether the following functions are differentiable at $x = 0$.

(a) $f(x) = \begin{cases} 1 + 3x - x^2, & \text{when } x < 0 \\ x^2 + 3x + 2, & \text{when } x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

(c) $f(x) = |\sin x|$

(d) $f(x) = x|x|$

Solution

(a) Note that

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x^2+3x+2\\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} 1+3x-x^2\\ &= \lim_{x\to 0^-} 1 \neq 2\end{aligned}$$

Hence, f is not continuous at $x=0$, thus not differentiable at $x=0$.

(b)

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}\\ &= \lim_{y\to\infty} ye^{-y^2} \quad (\text{Let } y=\frac{1}{x})\\ &= \lim_{y\to\infty}\frac{y}{e^{y^2}}\\ &= \lim_{y\to\infty}\frac{1}{y}\frac{y^2}{e^{y^2}}\\ &= \left(\lim_{y\to\infty}\frac{1}{y}\right)\left(\lim_{y\to\infty}\frac{y^2}{e^{y^2}}\right)\\ &= 0\cdot 0 = 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^-}\frac{e^{-\frac{1}{x^2}}}{x}\\ &= \lim_{y\to-\infty} ye^{-y^2} \quad (\text{Let } y=\frac{1}{x})\\ &= \lim_{y\to-\infty}\frac{y}{e^{y^2}}\\ &= \lim_{y\to-\infty}\frac{1}{y}\frac{y^2}{e^{y^2}}\\ &= \left(\lim_{y\to-\infty}\frac{1}{y}\right)\left(\lim_{y\to-\infty}\frac{y^2}{e^{y^2}}\right)\\ &= \left(\lim_{y\to-\infty}\frac{1}{y}\right)\left(\lim_{y^2\to\infty}\frac{y^2}{e^{y^2}}\right)\\ &= 0\cdot 0 = 0\end{aligned}$$

Hence, f is differentiable at $x=0$.

(c)

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{|\sin x|-0}{x}\\ &= \lim_{x\to 0^+}\frac{\sin x}{x}\\ &= 1\end{aligned}$$

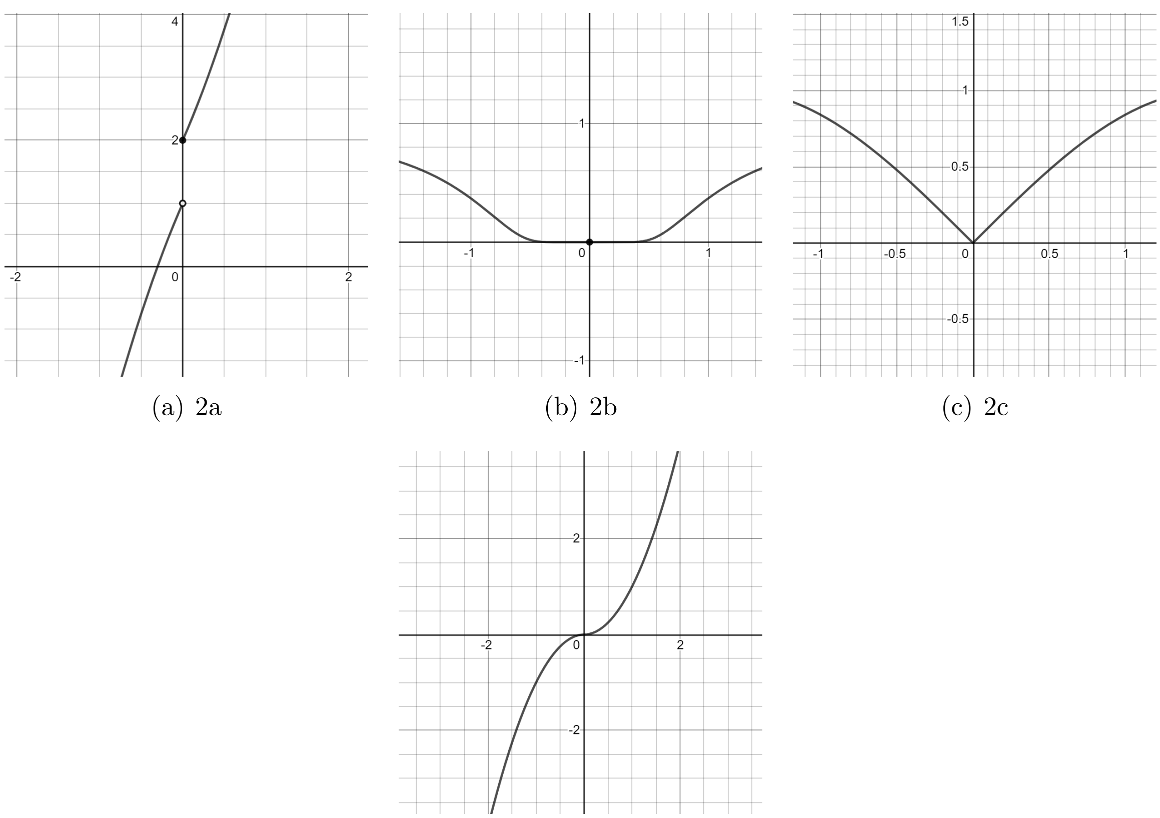
$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{|\sin x|-0}{x}$$
$$= \lim_{x\to 0^-}\frac{-\sin x}{x}$$
$$= -1 \neq 1$$

Hence, f is not differentiable at $x=0$.

(d)

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x|x|-0}{x}$$
$$= \lim_{x\to 0^+}\frac{x^2}{x}$$
$$= 0$$

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{x|x|-0}{x}$$
$$= \lim_{x\to 0^-}\frac{-x^2}{x}$$
$$= 0$$



(a) 2a (b) 2b (c) 2c

(d) 2d

Figure 1: Graph of Q2

3. Let $f(x) = |x|^3$.

(a) Find $f'(x)$ for $x \neq 0$.

(b) Show that $f(x)$ is differentiable at $x = 0$.

(c) Determine whether $f'(x)$ is differentiable at $x = 0$.

Solution

(a)

$$f'(x) = \begin{cases} 3x^2, & \text{when } x > 0; \\ -3x^2, & \text{when } x < 0. \end{cases}$$

(b) Note that

$$\lim_{h \to 0} \frac{|h|^3 - 0}{h - 0} = \lim_{h \to 0} \frac{|h|h^2}{h} = \lim_{h \to 0} |h|h = 0.$$

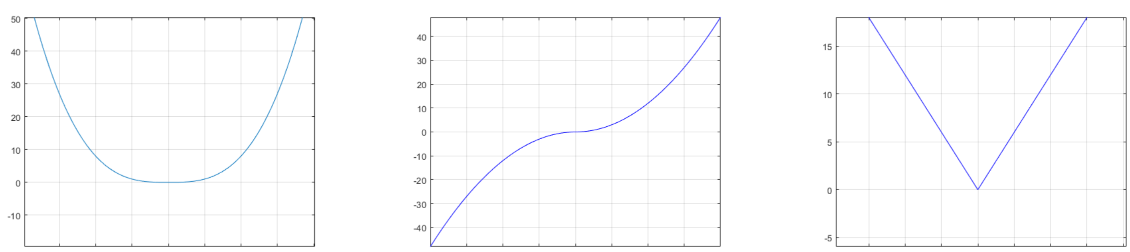
Hence f is differentiable at $x = 0$ with $f'(x) = 0$.

(c) Note that, by (a) and (b),

$$\lim_{h \to 0^+} \frac{f'(h) - f'(0)}{h - 0} = \lim_{h \to 0^+} \frac{3h^2}{h} = \lim_{h \to 0^+} 3h = 0.$$

$$\lim_{h \to 0^-} \frac{f'(h) - f'(0)}{h - 0} = \lim_{h \to 0^-} \frac{-3h^2}{h} = \lim_{h \to 0^-} -3h = 0.$$

Hence $f'(x)$ is differentiable at $x = 0$ with $f''(x) = 0$.



(a) graph of f (b) graph of f' (c) graph of f''

Figure 2: Graph of Q3

4. Let

$$f(x) = \begin{cases} (x-1)^2 \sin\left(\dfrac{1}{x-1}\right), & \text{when } x \neq 1; \\ 0, & \text{when } x = 1. \end{cases}$$

(a) Is f continuous on $\mathbb{R}$?

(b) Is f differentiable on $\mathbb{R}$?

(c) Is f' continuous on $\mathbb{R}$?

Solution

(a) $\lim_{x\to 1} f(x)$

$= \lim_{x\to 1}(x-1)^2 \sin\left(\dfrac{1}{x-1}\right)$

$= 0$ (by squeeze theorem)

$= f(1)$

So f is continuous.

(b) $\lim_{x\to 1} \dfrac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1}(x-1)\sin\left(\dfrac{1}{x-1}\right)$

$= 0$ by squeeze theorem.

So $f'(1) = 0$.

When $x \neq 2$,

$f'(x)$

$= \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{1}{h}\left((x+h-1)^2 \sin\left(\dfrac{1}{x+h-1}\right) - (x-1)^2 \sin\left(\dfrac{1}{x-1}\right)\right)$

$= \lim_{h\to 0} \dfrac{1}{h}\left((x-1)^2\left(\sin\left(\dfrac{1}{x+h-1}\right) - \sin\left(\dfrac{1}{x-1}\right)\right) + (2h(x-1)+h^2)\sin\left(\dfrac{1}{x+h-1}\right)\right)$

$= \left[\lim_{h\to 0} \dfrac{1}{h}(x-1)^2\left(2\cos\left(\dfrac{x-1+h/2}{(x+h-1)(x-1)}\right)\sin\left(\dfrac{-h/2}{(x+h-1)(x-1)}\right)\right)\right] +$

$(x-1)\sin\left(\dfrac{1}{x-1}\right)$

$= -\cos\left(\dfrac{1}{x-1}\right) + (x-1)\sin\left(\dfrac{1}{x-1}\right)$

So f is differentiable.

(c) $\lim_{x\to 1} f'(x)$ does not exist. So f' is not continuous.

5. Find natural domains of the following functions and differentiate them on their natural domains. You are not required to do so from first principles.

   (a) $f(x) = \dfrac{\sin\, x}{1 + \cos\, x}$.
   (b) $f(x) = (1 + \tan^2 x) \cos^2 x$.
   (c) $f(x) = \ln(\ln(\ln\, x))$
   (d) $f(x) = \ln|\sin\, x|$

Solution

(a)

$$\begin{aligned} 1 + \cos x &= 0 \\ \cos x &= -1 \\ x &= (2n-1)\pi, n \in \mathbb{Z} \end{aligned}$$

Therefore, the natural domain is $\mathbb{R} \setminus \{(2n-1)\pi : n \in \mathbb{Z}\}$.

$$\begin{aligned} f'(x) &= \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \\ &= \frac{\cos x + 1}{(1+\cos x)^2} \\ &= \frac{1}{1+\cos x} \end{aligned}$$

(b) $\tan x$ is well-defined on $\mathbb{R} \setminus \{\frac{(2n-1)\pi}{2} : n \in \mathbb{Z}\}$. Therefore, this is also the natural domain of f.
Note that $f(x) = (1+\tan^2 x)\cos^2 x = \cos^2 x + \sin^2 x = 1$. Hence, $f'(x) = 0$.

(c)

$$\begin{aligned} \ln x &> 0 && (1) \\ x &> 1 && (2) \\ \ln(\ln x) &> 0 && (3) \\ \ln x &> 1 && (4) \\ x &> e && (5) \end{aligned}$$

By considering the intersection of the intervals above, the natural domain is given by (e, ∞).

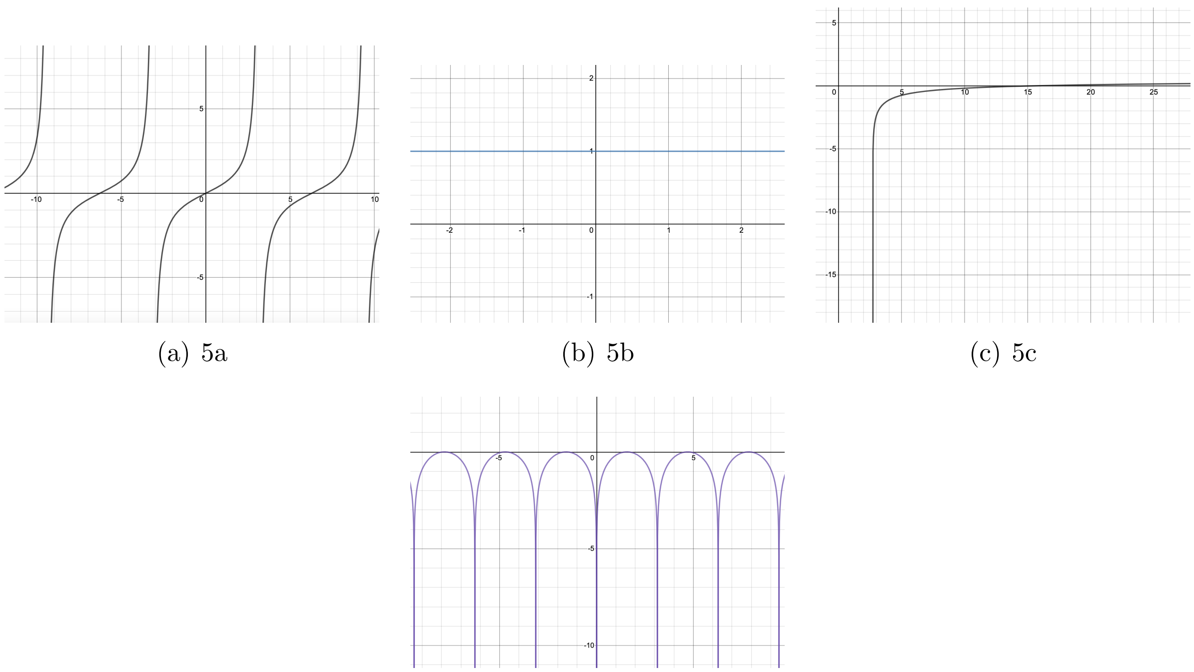
$$\begin{aligned} f'(x) &= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ &= \frac{1}{x \ln x \ln(\ln x)} \end{aligned}$$

(d)

$$
\begin{aligned}
|\sin x| &> 0 \\
\sin x &\neq 0 \\
x &\neq n\pi, n \in \mathbb{Z}
\end{aligned}
$$

Therefore, the natural domain of f is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$. Note that $f(x) = \ln(\pm \sin x)$. Therefore,

$$
\begin{aligned}
f'(x) &= \frac{1}{\pm \sin x} \cdot \pm \cos x \\
&= \frac{\cos x}{\sin x} \\
&= \cot x
\end{aligned}
$$



(a) 5a

(b) 5b

(c) 5c

(d) 5d

Figure 3: Graph of Q5

6. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose f is differentiable at $x = 0$, with $f'(0) = a$. Show that $f(x) = ax$.

Solution

Let $x = y = 0$, we have

$$f(0) = 2f(0).$$

Hence $f(0) = 0$.

Since f is differentiable at $x = 0$, we have

$$a = f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}.$$

For each fixed $x \in \mathbb{R}$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) + f(h) - f(x)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = a.$$

This indicates that f is differentiable everywhere with $f'(x) = a$. Then $f(x) = ax+c$ for some $c \in \mathbb{R}$.
However, we must have $c = 0$ since $f(0) = c = 0$.

7. Find $\dfrac{dy}{dx}$ if

(a) $x^2 + y^2 = e^{xy}$

(b) $x^3y + \sin xy^2 = 1$

(c) $y = \tan^{-1}\sqrt{x}$

(d) $y = 2^{\sin x}$

(e) $y = x^{\ln x}$

(f) $y = x^{x^x}$

Solution

(a) $x^2 + y^2 = e^{xy}$

$$2x + 2y\frac{dy}{dx} = \left(1 + x\frac{dy}{dx}\right)e^{xy}$$

$$\frac{dy}{dx} = \frac{e^{xy} - 2x}{2y - xe^{xy}}$$

(b) $x^3y + \sin xy^2 = 1$

$$3x^2y + x^3\frac{dy}{dx} + \left(y^2 + 2xy\frac{dy}{dx}\right)\cos xy^2 = 0$$

$$\frac{dy}{dx} = \frac{-3x^2y - y^2\cos xy^2}{x^3 + 2xy\cos xy^2}$$

(c) $y = \tan^{-1}\sqrt{x}$

$$\tan y = \sqrt{x}$$

$$\sec^2 y\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{\cos^2 y}{2\sqrt{x}}$$

(d) $y = 2^{\sin x}$

$$\frac{dy}{dx} = 2^{\sin x}\ln 2\cos x$$

(e) $y = x^{\ln x}$

$$\ln y = (\ln x)^2$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\ln x}{x}$$

$$\frac{dy}{dx} = \frac{2y\ln x}{x}$$

(f) $y = x^{x^x}$

$\ln y = x^x \ln x$

$\ln\ln y = x\ln x + \ln\ln x$

$\dfrac{1}{y\ln y}\dfrac{dy}{dx} = \ln x + 1 + \dfrac{1}{x\ln x}$

$\dfrac{dy}{dx} = (y\ln y)\left(\ln x + 1 + \dfrac{1}{x\ln x}\right)$

8. Find $\dfrac{d^2y}{dx^2}$ if

   (a) $y = \ln\tan x$

   (b) $y = \sin^{-1}\sqrt{1-x^2}$

Solution

(a)

$$\frac{dy}{dx} = \frac{1}{\tan x}\cdot\sec^2 x = \frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x} = \frac{1}{\sin x\cos x} = \frac{2}{\sin 2x} = 2\csc(2x)$$

$$\frac{d^2y}{dx^2} = -4\csc(2x)\cot(2x)$$

(b)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}}\cdot\frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{x^2-x^4}}$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{x^2-x^4} - x\cdot\frac{2x-4x^3}{2\sqrt{x^2-x^4}}}{x^2-x^4} = -\frac{x^2-x^4-x(x-2x^3)}{(x^2-x^4)^{\frac{3}{2}}} = -\frac{x^4}{(x^2-x^4)^{\frac{3}{2}}}$$

9. Find the n-th derivative of the following functions for all positive integers n.

   (a) $f(x) = (e^x + e^{-x})^2$, $x \in \mathbb{R}$

   (b) $f(x) = \dfrac{1}{1-x^2}$, $x \in (-1, 1)$

   (c) $f(x) = \sin x \cos x$, $x \in \mathbb{R}$

   (d) $f(x) = \cos^2 x$, $x \in \mathbb{R}$

   (e) $f(x) = \dfrac{x^2}{e^x}$, $x \in \mathbb{R}$

Solution

(a) Simplify $f(x)$ first,

$$f(x) = (e^x + e^{-x})^2 = e^{2x} + 2 + e^{-2x}.$$

Hence,

$$f^{(n)}(x) = 2^n e^{2x} + (-2)^n e^{-2x}.$$

(b) Process the partial fraction for $f(x)$. Suppose

$$f(x) = \frac{A}{1+x} + \frac{B}{1-x},$$

where A, B is a constant, then we have

$$\frac{1}{1-x^2} = \frac{(B-A)x + (B+A)}{1-x^2},$$

by comparing the coefficients, we have

$$\begin{cases} B + A &= 1, \\ B - A &= 0. \end{cases}$$

Hence, $A = B = \frac{1}{2}$, and

$$f(x) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right).$$

Therefore,

$$f^{(n)}(x) = \frac{1}{2}\left[(-1)^n \frac{n!}{(1+x)^{n+1}} + \frac{n!}{(1-x)^{n+1}}\right].$$

(c) By double angle formula,

$$f(x) = \sin x \cos x = \frac{1}{2}\sin 2x.$$

Hence,

$$f^{(n)}(x) = \begin{cases} 2^{n-1}\sin 2x & \text{if } n = 4k \text{ for some } k \in \mathbb{N}, \\ 2^{n-1}\cos 2x & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\sin 2x & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\cos 2x & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{N}. \end{cases}$$

(d) By double angle formula,

$$f(x) = \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Hence,

$$f^{(n)}(x) = \begin{cases} 2^{n-1}\cos 2x & \text{if } n = 4k \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\sin 2x & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\cos 2x & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{N}, \\ 2^{n-1}\sin 2x & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{N}. \end{cases}$$

(e) Note that

$$f(x) = \frac{x^2}{e^x} = x^2e^{-x} = g(x)h(x)$$

where $g(x) = x^2$, $h(x) = e^{-x}$. Using Leibniz Rule (proved by mathematical induction and product rule),

$$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(x)\, h^{(n-k)}(x).$$

Note that $g'(x) = 2x$, $g''(x) = 2$ and $g^{(k)}(x) = 0$ for all $k \geq 3$. Hence,

$$\begin{aligned} f^{(n)}(x) &= \binom{n}{0} g(x)\, h^{(n)}(x) + \binom{n}{1} g'(x)\, h^{(n-1)}(x) + \binom{n}{2} g''(x)\, h^{(n-2)}(x) \\ &= (-1)^n x^2 e^{-x} + (-1)^{n+1} 2nxe^{-x} + (-1)^n n(n-1)e^{-x}. \end{aligned}$$

10. Find all points (x_0, y_0) on the graph of

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 8$$

where lines tangent to the graph at (x_0, y_0) have slope -1.

Solution

We differentiate both sides of the equation and get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0.$$

Thus,

$$y' = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

Since $y' = -1$ at (x_0, y_0), we have

$$y_0^{\frac{1}{3}} = x_0^{\frac{1}{3}},$$

and thus $x_0 = y_0$. Plugging this back to the equation, we have

$$2x_0^{\frac{2}{3}} = 8,$$

and so $x_0 = \pm 8$. Therefore, $(x_0, y_0) = (8, 8)$ or $(-8, -8)$.

11. The chain rule says

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

or equivalently,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $y = f(u)$ and $u = g(x)$.

(a) Give examples to show

$$(f \circ g)''(x) \neq f''(g(x)) \cdot g''(x),$$

or equivalently,

$$\frac{d^2y}{dx^2} \neq \frac{d^2y}{du^2} \cdot \frac{d^2u}{dx^2},$$

where $\dfrac{d^2y}{dx^2}$ denotes the second derivative of $y = f(x)$.

(b) Prove that

$$(f \circ g)''(x) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).$$

Solution

(a) Let $y = u^2$ and $u = x$.
Then $y = x^2$.
$\dfrac{dy}{dx} = 2x$
$\dfrac{d^2y}{dx^2} = 2$
$\dfrac{d^2u}{dx^2} = 0$
$\dfrac{d^2y}{du^2} \cdot \dfrac{d^2u}{dx^2} = 0$

(b) $y = f(u)$ and $u = g(x)$.
$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\dfrac{dy}{dx}$
$= \dfrac{d}{dx}\left(\dfrac{dy}{du} \cdot \dfrac{du}{dx}\right)$
$= \dfrac{d}{dx}\left(\dfrac{dy}{du}\right) \cdot \dfrac{du}{dx} + \dfrac{dy}{du} \cdot \dfrac{d^2u}{dx^2}$
$= \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du} \cdot \dfrac{d^2u}{dx^2}$

12. (a) Suppose $a, b > 0$ are constants, and

$$y = \frac{1}{ab} \arctan\left(\frac{b}{a} \tan x\right)$$

for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Express $\dfrac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

(b) Suppose $a, b > 0$ are constants, and

$$y = \ln\left|\frac{a + b\tan x}{a - b\tan x}\right|$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left\{\pm \arctan \frac{b}{a}\right\}$. Express $\frac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

Solution

(a)

$$\frac{dy}{dx} = \frac{1}{ab}\frac{1}{1+(\frac{b}{a}\tan x)^2} \cdot \frac{b}{a}\sec^2 x = \frac{1}{a^2\cos^2 x + b^2 \sin^2 x}$$

(b) Note that

$$y = \ln(|\frac{a\cos x + b\sin x}{a\cos x - b\sin x}|) = \ln(\pm\frac{a\cos x + b\sin x}{a\cos x - b\sin x})$$

$$\begin{aligned}\frac{dy}{dx} =& (\pm\frac{a\cos x - b\sin x}{a\cos x + b\sin x})\times \\ & (\pm\frac{(a\cos x - b\sin x)(-a\sin x + b\cos x) - (a\cos x + b\sin x)(-a\sin x - b\cos x)}{(a\cos x - b\sin x)^2}) \\ =& \frac{2ab}{a\cos^2 x - b\sin^2 x}\end{aligned}$$