Math 2230A, Complex Variables with Applications

1. Use established properties of moduli to show that when $|z_3| \neq |z_4|$
$$\frac{Re(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$
.

2. Verify that $\sqrt{2}|z| \geq |Rez| + |Imz|$.
Suggestion:Reduce this inequality to $(|x|-|y|)^2 \geq 0$.

3. In each case, sketch the set of points determined by the given condition:
   (a) $|z-1+i| = 1$;
   (b) $|z+i| \leq 3$;
   (c) $|z-4i| \geq 4$.

4. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that $|z-1| = |z+i|$ represents the line through the origin whose slope is -1.

5. Use properties of conjugates and moduli established in Sec.6 to show that
   (a) $\overline{\bar{z}+3i} = z - 3i$;
   (b) $\overline{iz} = -i\bar{z}$;
   (c) $\overline{(2+i)^2} = 3-4i$;
   (d) $|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}|2z+5|$.

6. Sketch the set of points determined by the condition
   (a) $Re(\bar{z}-i) = 2$;
   (b) $|2\bar{z}+i| = 4$.

7. Show that
$$|Re(2+\bar{z}+z^3)| \leq 4 \quad when \quad |z| \leq 1.$$

8. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (2), Sec. 5, show that if z lies on the circle $|z| = 2$, then
$$|\frac{1}{z^4-4z^2+3}| \leq \frac{1}{3}.$$

9. Find the principal argument Argz when
(a) $z = \frac{-2}{1+\sqrt{3}i}$; (b)z=$(\sqrt{3}-i)^6$.

10. Show that (a) $|e^{i\theta}| = 1$; (b) $\overline{e^{i\theta}} = e^{-i\theta}$.

11. Using the fact that the modules $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1(see Sec.4), give a geometric argument to find a value of θ in the interval $0 \leq \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1| = 2$.

12. By writing the individual factors on the left in exponential form, perform the needed operations, and finally changing back to rectangular coordinates, show that

    (a) $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$;

    (b) $5i/(2 + i) = 1 + 2i$;

    (c) $(\sqrt{3} + i)^6 = -64$;

    (d) $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$.

13. Let z be a nonzero complex number and n a negative integer (n=$-1, -2, ...$). Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, ...$ Using the expressions

$$z^m = r^m e^{im\theta} \quad and \quad z^{-1} = (\frac{1}{r})e^{i(-\theta)},$$

verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec.7 could have been written alternatively as $z^n = (z^m)^{-1}$.

14. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1)$$

and use it to derive ***Lagrange's trigonometric identity:***

$$1 + cos\theta + cos2\theta + \cdots + cosn\theta = \frac{1}{2} + \frac{sin[(2n+1)\theta/2]}{2sin(\theta/2)} \quad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write S=$1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

15. Find the square roots of (a) $2i$;(b) $1-\sqrt{3}i$ and express them in rectangular coordinates.

16. Find the three cube roots $c_k(k = 0, 1, 2)$ of -8i, express them in rectangular coordinates, and point out why they are as shown in Fig.12.

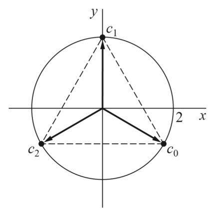

FIGURE 12

17. Find $(-8-8\sqrt{3}i)^{\frac{1}{4}}$, express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out which is the principal root.

18. In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root: $(a)(-1)^{\frac{1}{3}}$; $\quad (b)8^{\frac{1}{6}}$.

19. Find the square roots of $(a)2i;(b)1-\sqrt{3}i$ and express them in rectangular coordinates.

20. In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root: $(a)(-1)^{\frac{1}{3}}$; $\quad (b)8^{\frac{1}{6}}$.