MATH 2230A - HW 9 - Solutions

Full solutions at P.242 Q1, P.247 Q3, P.253. Q1

Commonly missed steps in Purple and common mistakes at the back; P.247 Q3b is important

Below are the key definition and theorems related to the HW.

Definition 0.1. Let $z_0 \in \mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function defined on a domain (an open connected set). Suppose f is analytic on a deleted neighborhood of z_0, that is $B(z_0, r)\backslash\{z_0\}$. Then f is its (unique) Laurentz Series at z_0 on the deleted neighborhood: for all $z \in B(z_0, r)\backslash\{z_0\}$, we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C}$ (and are unique). Then

- We call a_1 the residue of f at z_0 and denote it by $\text{Res}(f, z_0)$.
- We call $\sum_{n=-1}^{-\infty} a_n (z - z_0)^n$ the principal part of f (consisting only of negative power)
- Suppose the principal part of f at z_0 is nonzero.We call z_0 an essential singularity if it is inifinitely many terms. We call z_0 a pole of order $m \in \mathbb{N}$ if the principal part only has finitely many terms with the greatest negative power being m. A pole of order 1 is called a simple pole.
- Suppose the principal part of f at z_0 is zero. Then we call z_0 a removable singularity. In this case, the Laurentz Series at z_0 reduces to a Taylor Series: we can extend f to an analytic function at z_0 by defining $f(z_0)$ to be a_0, the constant term of the Taylor Series.
- Now suppose z_0 is a removable singularity and f is extended to take values at z_0. We call z_0 a zero of f if $f(z_0) = 0$. We say z_0 is a zero of order $m \in \mathbb{N}$ if a_m is the first non-zero coefficient in the Taylor Series. (If no such $m \in \mathbb{N}$ exists, then it means essentially that the Taylor series is constant zero at z_0, so f is locally *zero* at z_0.)

Proposition 0.2 (Basic Properties for Poles and Zeros)**.** *Let $z_0 \in \mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function defined on a domain. Suppose f is analytic on a deleted neighborhood of z_0. Then we have*

1. *z_0 is a zero of order $m \in \mathbb{N}$ there exists a function g analytic at z_0 and $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^m g(z)$ locally , that is, for some neighborhood of z_0.*
2. *z_0 is a pole of order $m \in \mathbb{N}$ if and only if there exists a function g analytic at z_0 and $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^{-m} g(z)$ locally, that is, for some neighborhood of z_0.*
3. *z_0 is a zero of f of order $m \in \mathbb{N}$ if and only if z_0 is a pole of $\frac{1}{f}$ of order m.*
4. *z_0 is a zero of order m if and only if $f^{(k)}(z_0) = 0$ for all $k < m$, but $f^{(m)}(z_0) \neq 0$*

Remark. It is a good exercise to verify all the above properties yourself.

Theorem 0.3 (Isolation of Zeros)**.** *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a domain. Suppose f has a zero at $a \in \Omega$, that is, $f(a) = 0$. Then there exists a neighborhood $B(a, r)$ of a such that either $f(z) = 0$ for all $z \in B(a, r)$ or $f(z) \neq 0$ for all $z \in B(a, r)\backslash\{z\}$.*

Remark. This is easily proven from Taylor Series. In fact by the *connectedness* of domain, one can strength the result to that either f is constantly 0 on Ω or f only can have isolated zeros on Ω

Proposition 0.4 (Computation of Residues for simple pole)**.** *Let $f : \Omega \to \mathbb{C}$ be analytic on a deleted neighborhood of z_0. Suppose $z_0 \in \Omega$ is a pole of order $m \in \mathbb{N}$. Then the residue of f at z_0 is given by*

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \left.\frac{d^{m-1}}{dz^{m-1}}\right|_{z=z_0} g(z)$$

where $g(z) = (z - z_0)^m f(z)$ locally (in a neighborhood) at z_0 and is holomorphic nonzero at z_0.

Remark. You do not have to memorize this result: just think about the Laurent Series of f at z_0 then this result is clear. Moreover, we often only use this result when $m = 1$. In that case, we do not even have to compute derivatives.

Theorem 0.5 (Residue Theorem)**.** *Let $f : \Omega \to \mathbb{C}$ be a function defined on a simply connected domain. Suppose f is analytic on $\overline{\Omega}$ except maybe for a finite number of points $\{z_n\}_{n=1}^k \subset \Omega$. Then we have*

$$\int_{\partial\Omega} f(z)dz = \sum_{n=1}^{k} \text{Res}(f; z_n)$$

Remark. The Residue Theorem basically follows when you try to integrate the Laurentz Series at each of the singularities along some circles around them (by the Cauchy Theorem on multiply connected domain). The Taylor Series parts are integrated to 0 as they are holomophic (by Cauchy-Goursat Theorem) while the non-simple parts in the Principal parts (i.e. the part with negative powers) vanish except for the $1/z$ term because anti-derivatives exist (by Independence of Paths).

Example 0.6. *Here are some commonly used Taylor Series (centered at $z_0 := 0$):*

a) $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots;\ z \in \mathbb{C}$

b) $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \ldots;\ z \in \mathbb{C}$

c) $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - z^2 + \frac{z^4}{4!} - \ldots;\ z \in \mathbb{C}$

d) $\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \ldots;\ z \in \mathbb{C}$

e) $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + z^2 + \frac{z^4}{4!} + \ldots;\ z \in \mathbb{C}$

P.242

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

(a) $z\exp\left(\dfrac{1}{z}\right)$; (b) $\dfrac{z^2}{1+z}$; (c) $\dfrac{\sin z}{z}$; (d) $\dfrac{\cos z}{z}$; (e) $\dfrac{1}{(2-z)^3}$.

Solution. 1. Let $f(z) = z\exp(1/z)$. The f is analytic except at $z = 0$ so $z = 0$ is an isolated singularity of f. When $z \neq 0$, by Taylor series of the exponential function, we have $f(z) = z\exp\left(\frac{1}{z}\right) = z\sum_{k=0}^{\infty}\frac{1}{k!z^k} = \sum_{k=0}^{\infty}\frac{1}{k!z^{k-1}}$. By Uniqueness of Laurent Series, this is its Laurent Series. Hence, the principal part of f is $\sum_{k=2}^{\infty}\frac{1}{k!z^{k-1}}$. Since the principal part consists of infinitely many terms, 0 is an essential singularity.

2. Let $f(z) = \frac{z^2}{1+z}$. Then f is analytic except at $z = -1$ so $z = -1$ is an isolated singularity. When $z \neq -1$, we have $f(z) = \frac{((z+1)-1)^2}{z+1} = \frac{1}{1+z} - 2 + \frac{1}{1+z}$, which is the Laurent series of f at $z = -1$ (by uniqueness). Hence the principal part of f at -1 is $\frac{1}{1+z}$, which implies -1 is a simple pole of f.

3. Let $f(z) = \frac{\sin z}{z}$. Then f is analytic except at $z = 0$ so $z = 0$ is an isolated singularity. When $z \neq 0$, we have $f(z) = \sum_{k=0}^{\infty}\frac{1}{z}\frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k+1)!}$, which is the Laurent series of f at $z = -1$ (by uniqueness). Hence the principal part of f at 0 is 0, which implies 0 is a removable singularity of f.

4. Let $f(z) = \frac{\cos z}{z}$. Then f is analytic except at $z = 0$ so $z = 0$ is an isolated singularity. When $z \neq 0$, we have $f(z) = \sum_{k=0}^{\infty}\frac{1}{z}\frac{(-1)^k z^{2k}}{(2k)!} = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k-1}}{(2k)!}$, which is the Laurent series of f at $z = -1$ (by uniqueness). Hence the principal part of f at 0 is given by $\frac{1}{z}$, which implies 0 is a simple of f.

5. Let $f(z) = \frac{1}{(2-z)^3}$. Then f is analytic except at $z = 2$ so $z = 2$ is an isolated singularity. When $z \neq 2$, we have $f(z) = \frac{1}{(2-z)^3} = \frac{-1}{(z-2)^3}$, which is the Laurent series of f at $z = 2$ (by uniqueness). Hence the principal part of f at 2 is given by $\frac{-1}{(z-2)^3}$, which implies 2 is a pole of f.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; (b) $\dfrac{1-\exp(2z)}{z^4}$; (c) $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans. (a) $m = 1, B = -1/2$; (b) $m = 3, B = -4/3$; (c) $m = 2, B = 2e^2$.

Solution. Only the last one is a bit tricky: write $\exp(2z)$ as $\exp(2z-2)\exp(2)$. Then use the Taylor Series for the exponential function.

3. Suppose that a function f is analytic at z_0, and write $g(z) = f(z)/(z - z_0)$. Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in Sec. 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Solution. It is clear after writing out the Taylor series of f at z_0: $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$ for all z in some neighborhood of z_0.

P.247

3. In each case, find the order m of the pole and the corresponding residue B at the singularity $z = 0$:

$$(a)\ \frac{\sinh z}{z^4}; \qquad (b)\ \frac{1}{z(e^z - 1)}.$$

Ans. $(a)\ m = 3,\ B = \frac{1}{6}; \quad (b)\ m = 2,\ B = -\frac{1}{2}.$

Solution. (a) Let $f(z) = \frac{\sinh z}{z^4}$. Then $f(z) = \frac{1}{z^4}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} = \frac{1}{z^4}(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots)$ locally at $z = 0$. It is clear that $z = 0$ is a pole of order 3 and $\mathrm{Res}(f, 0) = \frac{1}{3!} = \frac{1}{6}$

(b) Let $f(z) = \frac{1}{z(e^z-1)}$. Let $g(z) = e^z - 1$. Note that $g(0) = 0$ but $g'(0) = 1 \neq 0$. Hence, 0 is a zero of order 1 of g. It is clear that 0 is a order-1 zero of $z \mapsto z$. Hence $z(e^z - 1)$ has a zero of order 2 at 0 which implies f has a pole of order 2 at 0. By the Residue Formula that follows readily by considering Laurentz Series (Proposition 0.4 in this Solution, or Formula 1.117 in Lecture Note), we have

$$\mathrm{Res}(f, 0) = \frac{d}{dz}\bigg|_{z=0} \frac{h(z)}{1!} = h'(0)$$

where $h(z) = z^2 f(z) = \frac{z}{e^z-1}$ locally at 0 and is holomorphic non-zero at 0. Note $h(z)(e^z-1) = z$. By differentiating both sides, we have $h'(z)(e^z - 1) + h(z)e^z = 1$, which implies $h(0) = 1$. Differentiating once more, we have $h''(z)(e^z-1)+h'(z)e^z+h'(z)e^z+h(z)e^z = 0$, which imples $2h'(0) + h(0) = 0$ and $h'(0) = -1/2$. Therefore, $\mathrm{Res}(f, 0) = -1/2$.

Remark. This technique for computing residue is standard. Please revise it. I will include some more computational techniques in the solution to HW 10.

4. Find the value of the integral

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)}\, dz,$$

taken counterclockwise around the circle $(a)\ |z - 2| = 2$; $(b)\ |z| = 4$.

Ans. $(a)\ \pi i; \quad (b)\ 6\pi i.$

Solution. (a) Use the Cauchy Integral Formula with $g(z) = \frac{3z^3+2}{z^2+9}$ and $z_0 = 1$.

(b) Let $f(z) = \frac{3z^3+2}{(z-1)(z^2+9)}$. Then $1, \pm 3i$ are isolated singularities within the simply connected contour, where f is holomorphic on whose closure. It follows by the Residue Theorem that $\int_C f(z)dz = 2\pi i(\mathrm{Res}(f, 1) + \mathrm{Res}(f, 3i) + \mathrm{Res}(f, -3i))$. Note that all the singularities are simple poles. Hence the computation for their residues are easy and is given by

a) $\mathrm{Res}(f, 1) = \sum_{z\to 1}(z - 1)f(z) = 1/2$

b) $\mathrm{Res}(f, 3i) = \sum_{z\to 3i}(z - 3i)f(z) = \frac{75i+2+243}{60i}$

c) $\mathrm{Res}(f, -3i) = \sum_{z\to 3i}(z - 3i)f(z) = \frac{75i-2-243}{60i}$

The result follows from this.

5. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

taken counterclockwise around the circle (*a*) $|z| = 2$; (*b*) $|z+2| = 3$.

Ans. (*a*) $\pi i/32$; (*b*) 0 .

Solution. (a) Use the generalized Cauchy integral formula with $g(z) = \frac{1}{z+4}$ at $z_0 = 0$.

(b) Let $f(z) = \frac{1}{z^3(z+4)}$By the Residue THeorem, it suffices to compute the Residue for $z = 0$ and $z = -4$. That latter is a simple pole so it is easy. We shall only show the computation of $\text{Res}(f, 0)$. Note that 0 is a pole of order 3 for f. Therefore, we have

$$\text{Res}(f, 0) = \frac{h^{(2)}(0)}{2!}$$

where $h(z) = z^3 f(z) = \frac{1}{z+4}$. The result follows clear.

Remark. You can also use the Cauchy Goursat Theorem for Multiply connected domain and the Cauchy integral Formula to do part b.

6. Evaluate the integral

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$$

when C is the circle $|z| = 2$, described in the positive sense.

Ans. $4\pi i$.

Solution. All singularities are simple poles. The computations is easy.

P.253

1. Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity by appealing to Theorem 2 in Sec. 83. (Compare with Exercise 3, Sec. 73, where this result is evident from a Laurent series.)

Solution. Let $f(z) = 1/\sin z$. Note that $\sin(0) = 0$ but $(\sin z)'(0) = \cos(0) = 1$. Therefore, 0 is a zero of order 1 for $z \mapsto \sin z$. So $f(z)$ has a simple pole at 0. By Theorem 2 in Sec. 83, it follows that $\text{Res}(f, 0) = 1/(\sin z)'(0) = 1$.

Remark. Actually the Theorem 2 in Sec.83 follows as well from the Residue Formula (Proposition 0.4 in this Solution, or Formula 1.117 in Lecture Note): since the singularity is a simple pole, the above residue can be computed by $\text{Res}(f, z) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z}{\sin z} = \lim_{z\to 0} \frac{1}{\cos z} = 1$ where the L'Hospital Rule is used in the 2nd last equality, which implies the result in the Theorem.

2. Use conditions (1) in Sec. 82 to show that the function

$$q(z) = 1 - \cos z$$

has a zero of order $m = 2$ at the point $z_0 = 0$.

Solution. Note that $q'(z) = \sin z$ and $q''(z) = \cos z$. Hence, we have $q(0) = q'(0) = 0$, but $q''(0) \neq 0$. By the condition, we conclude that q has a zero of order 2 at $z = 0$.