Math 2230A, Complex Variables with Applications

1. Find $(-8-8\sqrt{3}i)^{1/4}$, express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out which is the principal root.

2. In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:
(a) $(-1)^{1/3}$; (b) $8^{1/6}$.

3. In each case, sketch the closure of the set:
$(a) -\pi < \arg z < \pi (z \neq 0)$;
(b) $|\operatorname{Re} z| < |z|$;
$(c) \operatorname{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$;
$(d) \operatorname{Re}\left(z^2\right) > 0$.

4. Let S be the open set consisting of all points z such that $|z| < 1$ or $|z-2| < 1$. State why S is not connected.

5. Show that
(a) $\log(-ei) = 1 - \frac{\pi}{2}i$; (b) $\log(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

6. Show that
(a) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;
(b) $\log i = \left(2n + \frac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;
(c) $\log(-1+\sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

7. (a) Show that the two square roots of i are

$$e^{i\pi/4} \quad \text{and} \quad e^{i5\pi/4}.$$

Then show that

$$\log(e^{i\pi/4}) = \left(2n + \frac{1}{4}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(e^{i5\pi/4}) = \left[(2n+1) + \frac{1}{4}\right]\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Conclude that

$$\log(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) Show that

$$\log(i^{1/2}) = \frac{1}{2}\log i,$$

as stated in Example 5, Sec. 32, by finding the values on the right-hand side of this equation and then comparing them with the final result in part (a).

8. Show that
   (a) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$;
   (b) $\frac{1}{i^{2i}} = \exp[(4n+1)\pi] \quad (n = 0, \pm 1, \pm 2, \ldots)$.

9. Find the principal value of
   (a) $(-i)^i$; (b) $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$; (c) $(1-i)^{4i}$.

10. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(aln|z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

11. By using one of the identities (9) and (10) in Sec. 39 and then proceeding as in Exercise 15, Sec. 38, find all roots of the equation
    (a) $\sinh z = i$; (b) $\cosh z = \frac{1}{2}$.

12. Find all roots of the equation $\cosh z = -2$. (Compare this exercise with Exercise 16, Sec. 38.)