THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 9

Theorem 1. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any compact subset K of A, the Larrent series of f converges to f uniformly and absolutely for all $z \in K$*

Theorem 2. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any $a \in A$, we can differentiate the Larrent series of f term by term. That is,*

$$f'(a) = \sum_{n=1}^{\infty} n a_n (a - z_0)^{n-1} - \sum_{n=1}^{\infty} \frac{n b_n}{(a - z_0)^{n+1}}$$

Theorem 3. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any contour C inside A, we can integrate the Larrent series of f term by term. That is,*

$$\int_C f(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n dz + \sum_{n=1}^{\infty} b_n \int_C \frac{1}{(z - z_0)^n} dz$$

Remark : Theorem 2 and 3 are a immediate consequence of theorem 1.

Be careful that the contour in the above theorem may not be closed! If the contour is closed and contains z_0, we see that all the term are zero except the term $b_1 \int_C \frac{1}{(z-z_0)} dz$, it is because the terms $(z - z_0)^n$ have antiderivative in A except $\frac{1}{(z-z_0)}$ $(n = -1)$. This leads to an important theorem. Before that, we introduce some definitions.

Definition 1. *Suppose that f is analytic in some punctured disk $D = \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. The coefficient of $\dfrac{1}{(z - z_0)}$ in the Larrent series is called the residue of f at the singular point $z = z_0$, which is denoted by $\operatorname*{Res}_{z=z_k} f$. If we write $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \dfrac{b_n}{(z - z_0)^n}$, then $\operatorname*{Res}_{z=z_k} f = b_1$.*

Theorem 4. *(Cauchy Residue Theorem) Suppose C is a closed contour in positive sense. If f is analytic inside and on C except finite number of singular points z_k inside C, then*

$$\int_C f dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f$$

Remark : Actually it is exactly Cauchy integral formula in the view of power series.
Remark : In other words, to calculate the integral $\int_C f dz$ is to calculate the residue of f at the singular points.

Definition 2. *Suppose that f is analytic in some punctured disk $D = \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. We define the order of pole at z_0 to be the smallest non-negative integer m such that $\lim_{z \to z_0} f(z)(z - z_0)^{m+1} = 0$.*

Remark : If the order of pole at z_0 is m, it implies that the non-zero coefficients in Larrent series is at most up to b_m and $b_n = 0$ for all $n > m$. Since m is the smallest non-negative integer such that $\lim_{z\to z_0} f(z)(z-z_0)^{m+1} = 0$, if not, suppose that $b_{m+1} \neq 0$, we can see that $\lim_{z\to z_0} f(z)(z-z_0)^{m+1} = b_{m+1} \neq 0$ by expanding the Larrent series.

Then we come to the computation of residue. Of course we can express the whole Larrent series to obtain that. We provide an alternative method here. If the order of pole of f at $z = z_0$ is m and thus

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n}.$$

We consider

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^{n+m} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + ... + b_m$$

and differentiate it $m-1$ times, we could have

$$\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right) = (m-1)!b_1 + O(z-z_0)$$

Theorem 5. *Suppose that f is analytic in some punctured disk $D = \{z \in \mathbb{C} \mid 0 < |z-z_0| < R\}$ and the order of pole at z_0 is m, then* $\operatorname*{Res}_{z=z_k} f = \lim_{z\to z_0} \frac{1}{(m-1)!}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right)$.

Exercise:

1. Compute $\int_C e^{-\frac{1}{z}} dz$ where C representing the contour $\{|z| < 3\}$.
2. Compute $\int_C \frac{5z-2}{z(z-1)} dz$ where C representing the contour $\{|z| < 3\}$.
3. Compute $\int_C \frac{\pi}{z^2 \sin(\pi z)} dz$ where C representing the contour $\{|z| < \frac{1}{2}\}$.
4. Compute $\int_0^{\pi/2} \frac{d\theta}{a^2 + \sin^2\theta}$ for $a > 0$.