THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Spring MATH2230
Tutorial 6

Theorem 1. *Suppose that*

1. *C is simply closed contour in counterclockwise direction;*
2. *C_k(n=1,..,n) are simply closed contour interior to C, all in clockwise direction, that are disjoint and whose interiors have no common points.*

If f is analytic on all of the contour C and C_k and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then

$$\int_C f dz + \sum_1^n \int_{C_k} f dz = 0$$

Remark : You should draw a diagram about it.
Remark : $\int_C f dz = -\int_{-C} f dz$ where C is in counterclockwise direction and $-C$ is in clockwise direction.
Remark : You can replace the contour C with a circle or other "simple" contour in most of the case.

Theorem 2. *(Weak Cauchy-Goursat theorem) If $f(z)$ is analytic at all points interior to and on a simple closed contour C* ***except a point*** *z_0 interior to C and satisfies $\lim_{z \to z_0} |z - z_0| f(z) = 0$, then*

$$\int_C f(z) dz = 0$$

Remark : Such a point z_0 is called the removable singularity of f which will be taught later.
Remark : This theorem also holds for finitely many such a singularity.

Theorem 3. *(Cauchy Integral Formula) Let f be analytic inside and on a simple closed contour C. If z_0 is interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}$$

Remark : You may prove that by using Weak Cauchy-Goursat by setting $g = \dfrac{f(z) - f(z_0)}{z - z_0}$. By continuity of f we have $\lim_{z \to z_0} |z - z_0| g(z) = 0$.
Remark : You can see that an analytic function is uniquely determined by its boundary value. (compare with the case of real variable function)

Lemma 1. *Let h be continuous on a simple closed contour C. Define $H_n(z) = \int_C \dfrac{h(w)dw}{(w - z)^n}$ for $n \geq 1$ and z being inside the interior of C. Then H_n is analytic inside the interior of C and $H_n'(z) = nH_{n+1}(z)$.*

Using this lemma, we have:

Theorem 4. *(Generalized Cauchy Integral Formula) Let f be analytic inside and on a simple closed contour C. If z_0 is interior to C, then*

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

Remark : This is why analyticity implies complex infinite differentiability.

Exercise:

1. Find $\displaystyle\int_C \frac{dz}{z^2+4}$ where C represents the circle $|z-i| = 2$.
2. Find $\displaystyle\int_C \frac{dz}{(z-i/2)^5}$ where C represents the circle $|z-i| = 2$.
3. Find $\displaystyle\int_C \frac{dz}{(z^2+4)^2}$ where C represents the circle $|z-i| = 2$.
4. Find $\displaystyle\int_C \frac{\cos z dz}{z(z^2+8)}$ where C represents the square whose sides lie along $x = \pm 2$ and $y = \pm 2$.
5. Find $\displaystyle\int_C \frac{dz}{(z+1)^2(z^2+1)}$ where C represents the circle $|z| = 2$.
6. Find $\displaystyle\int_C \frac{e^{az}dz}{z} = 2\pi i$ where C represents the unit circle. And hence show that $\displaystyle\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi$