# Tutorial 12

**5.5.14** . Let $\mathcal{F}_R(t)$ be the Fejér kernel on the real line, i.e.

$$\mathcal{F}_R(t) = \begin{cases} R\left(\frac{\sin \pi tR}{\pi tR}\right)^2 & \text{if } t \neq 0, \\ R & \text{if } t = 0. \end{cases}$$

Let $F_N(t)$ be the Fejér kernel for 1-periodic functions.

that is,

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \frac{1}{N} \frac{\sin^2(N\pi x)}{\sin^2(\pi x)} .$$

Show that

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = F_N(x) .$$

*Proof.* By the Fourier inversion formula and the proof of Poisson summation formula, (replacing $\int_0^1 \left(\sum_{n=-\infty}^{\infty} f(x+n)\right) e^{-2\pi i n x} dx$ by $\int_0^1 \left(\sum_{n=-\infty}^{\infty} \hat{f}(\xi+n)\right) e^{2\pi i n \xi} d\xi$ ) one has

$$(*) \qquad \sum_{n=-\infty}^{\infty} \hat{f}(\xi+n) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi} .$$

Note that

$$\int_{-\infty}^{\infty} \chi_{[-N,N]}(x) \left(1 - \frac{|x|}{N}\right) e^{-2\pi i \xi x} dx$$

$$= \cdots = \mathcal{F}_N(\xi),$$ where $\chi_{[-N,N]}(x)$ is the characteristic function of $[-N,N]$

Therefore, define $f = \chi_{[-N,N]}(x)\left(1 - \frac{|x|}{N}\right)$

and then $(*)$ implies that

$$\sum_{n=-\infty}^{\infty} \hat{F}_N(x+n) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-2\pi i n x}$$

$$= \sum_{n=-N}^{N} \left(1-\frac{|n|}{N}\right) e^{-2\pi i n x}$$

$$= \sum_{n=-N}^{N} \left(1-\frac{|n|}{N}\right) e^{2\pi i n x} = F_N(x) \qquad \square$$

5.5.20 Suppose that $f$ is a function of moderate decrease and its Fourier transform $\hat{f}$ is supported in $I = [-\frac{1}{2}, \frac{1}{2}]$.

(a) Prove the the following reconstruction formula:

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\, K(x-n), \quad \text{where } K(y) = \frac{\sin \pi y}{\pi y}$$

(b) Let $\lambda > 1$. Show that

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) K_\lambda\left(x-\frac{n}{\lambda}\right)$$

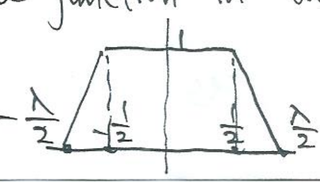
$$\text{where } K_\lambda(y) = \frac{\cos \pi y - \cos \pi \lambda y}{\pi^2 (\lambda - 1) y^2}$$

(c) Show that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |f(n)|^2$$

Hint: (a) Show that if $\chi_I$ is a characteristic function of $I$, then $\hat{f}(\xi) = \chi_I(\xi) \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}$.

(b) use the function in the following instead of $\chi_I$



Proof: (a) Applying Poisson summation formula to $\tilde{f}$, we have

$$\hat{f}(\xi) = \chi_I\left(\sum_{n=-\infty}^{\infty} \hat{f}(\xi+n)\right) = \chi_I\left(\sum_{n=-\infty}^{\infty} f(n)\, e^{-2\pi i n\xi}\right).$$

Note that $\displaystyle\int_{-\infty}^{\infty} \chi_I\, e^{2\pi i x\xi}\, d\xi = \frac{e^{\pi i x} - e^{-\pi i x}}{2\pi i x} = \frac{\sin \pi x}{\pi x}$.

Therefore,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi \overset{①}{=} \sum_{n=-\infty}^{\infty} f(n) \int_{-\infty}^{\infty} \chi_I\, e^{2\pi i (x-n)\xi}\, d\xi$$

① holds since $f \in \mathcal{M}(\mathbb{R})$

$\displaystyle\sum_{n=-\infty}^{\infty} f(n)\, \chi_I\, e^{2\pi i (x-n)\xi}$ converges absolutely

$$= \sum_{n=-\infty}^{\infty} f(n)\, K(x-n).$$

(b). Define $g(\xi) = \begin{cases} 1 & \text{if } |\xi| \le \frac{1}{2}, \\ \frac{2}{\lambda-1}\left(\frac{\lambda}{2} - |\xi|\right) & \text{if } \frac{1}{2} < |\xi| \le \frac{\lambda}{2}, \\ 0 & \text{if } \lambda > \frac{\lambda}{2}. \end{cases}$

Then $\displaystyle\int_{-\infty}^{\infty} g(\xi)\, e^{2\pi i x\xi}\, d\xi$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i x\xi}\, d\xi + \int_{\frac{1}{2}}^{\frac{\lambda}{2}} \frac{2}{\lambda-1}\left(\frac{\lambda}{2} - \xi\right) e^{2\pi i x\xi}\, d\xi + \int_{-\frac{\lambda}{2}}^{-\frac{1}{2}} \frac{2}{\lambda-1}\left(\frac{\lambda}{2} + \xi\right) e^{2\pi i x\xi}\, d\xi$$

$$= \cancel{\frac{e^{2\pi i x\xi}}{2\pi i x}\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}}} + \cancel{\frac{2}{\lambda-1}\left(\frac{\lambda}{2} - \xi\right)\frac{e^{2\pi i x\xi}}{2\pi i x}\Bigg|_{\frac{1}{2}}^{\frac{\lambda}{2}}} + \cancel{\frac{2}{\lambda-1}\left(\frac{\lambda}{2} + \xi\right)\frac{e^{2\pi i x\xi}}{2\pi i x}\Bigg|_{-\frac{\lambda}{2}}^{-\frac{1}{2}}}$$

$$+ \frac{2}{\lambda-1}\int_{\frac{1}{2}}^{\frac{\lambda}{2}} \frac{e^{2\pi i x\xi}}{2\pi i x}\, d\xi - \frac{2}{\lambda-1}\int_{-\frac{\lambda}{2}}^{-\frac{1}{2}} \frac{e^{2\pi i x\xi}}{2\pi i x\xi}\, d\xi$$

$$= \frac{2}{\lambda-1}\frac{1}{(2\pi i x)^2}\left(e^{\pi i x\lambda} + e^{-\pi i x\lambda} - e^{\pi i x} - e^{-\pi i x}\right)$$

$$= \frac{\cos \pi x - \cos \pi\lambda x}{\pi^2(\lambda-1)x^2} = K_\lambda(x)$$

~~Since~~ $\cancel{\hat{f}\left(\frac{\xi}{\lambda}\right) = g(\xi)\left(\sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{\xi}{\lambda}+n\right)\right)}$

$$\cancel{= g(\xi)}$$

Let $h(x) = f\left(\frac{x}{\lambda}\right)$, and then we have $\hat{h}(\xi) = \lambda \hat{f}(\lambda \xi)$ and $\hat{h}$ is supported in $\left[-\frac{1}{2\lambda}, \frac{1}{2\lambda}\right] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Hence,

$$\hat{h}(t) = g(\lambda t)\, \hat{h}(t) = g(\lambda t)\left(\sum_{n=-\infty}^{\infty} h(n)\, e^{-2\pi i n t}\right).$$

Substitute $\xi = \lambda t$ one has

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) g(\xi)\, e^{-2\pi i \frac{n}{\lambda}\xi}$$

Therefore, it follows from the Fourier inversion formula that

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) K_\lambda\left(x - \frac{n}{\lambda}\right).$$

(c) $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{n=-\infty}^{\infty} f(n)\, e^{-2\pi i n \xi} \right|^2 d\xi$$

$$= \sum_{n=-\infty}^{\infty} |f(n)|^2$$

$\square$