Tutorial 11

5.5.8. Prove that f is continuous, of moderate decrease, and $\int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy}\, dy = 0$ for all $x \in \mathbb{R}$, then $f = 0$.

Proof. Define $g = f * e^{-x^2}$, and then

$$g(x) = \int_{-\infty}^{\infty} f(y)\, e^{-(x-y)^2} dy = e^{-x^2} \int_{-\infty}^{\infty} f(y)\, e^{-y^2+2xy} dy = 0.$$

Therefore,

$$\hat{g}(\xi) = 0 = \hat{f}(\xi) \cdot \widehat{e^{-x^2}}(\xi) = \hat{f}(\xi) \cdot \sqrt{\pi}\, e^{-\pi^2 \xi^2}$$

$$\Rightarrow \hat{f}(\xi) = 0.$$

So it ~~is~~ is easy to obtain that $f = 0$ for all x

by the fourier inversion formula / Plancherel's identity !!! $\square$

Ex. Define $h(x) := e^{-|x|} \cos x$. Then $\hat{h}(\xi) = \dfrac{2(2\pi\xi)^2 + 4}{(2\pi\xi)^4 + 4}$.

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2}\, dx$.

Sol.

$$\int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2}\, dx = \int_{-\infty}^{\infty} \frac{1}{4} \left(\frac{2(2\pi\xi)^2+4}{(2\pi\xi)^4+4} \right)^2 d(2\pi\xi)$$

$$= \frac{\pi}{2} \int_{-\infty}^{\infty} e^{-2|x|} \cos^2 x\, dx$$

$$= \frac{\pi}{2} \int_{0}^{\infty} e^{-2x} (\cos 2x + 1)\, dx$$

$$= \frac{\pi}{4} \int_{0}^{\infty} e^{-x} (\cos x + 1)\, dx$$

$$\int_0^\infty e^{-x}\cos x\,dx = \left(-e^{-x}\cos x\right)\Big|_0^\infty - \int_0^\infty (-e^{-x})(-\sin x)\,dx$$

$$= 1 - \int_0^\infty e^{-x}\sin x\,dx$$

$$= 1 - \left[(-e^{-x})\sin x\right]\Big|_0^\infty + \int_0^\infty (-e^{-x})\cos x\,dx$$

$$\Rightarrow \int_0^\infty e^{-x}\cos x = \frac{1}{2}.$$

$$\Rightarrow \int_{-\infty}^\infty \frac{(x^2+2)^2}{(x^4+4)^2}\,dx = \frac{\pi}{4}\int_0^\infty e^{-x}(\cos x+1)\,dx = \frac{\pi}{4}\left(\frac{1}{2}+1\right)$$

$$= \frac{3}{8}\pi$$

$\square$

5.5.12 Define:

$$u(x,t) = \frac{x}{t}\mathcal{H}_t(x),$$

where $\mathcal{H}_t(x)$ is the heat kernel given by

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Show that: (a) u satisfies the heat eqn for $t>0$;

(b) $\lim_{t\to 0} u(x,t) = 0$ for every $x\in\mathbb{R}$;

(c) u is not continuous at the origin.

Proof. (a) $$\frac{\partial u}{\partial t} = -\frac{x}{t^2}\mathcal{H}_t(x) + \frac{x}{t}\frac{\partial \mathcal{H}_t(x)}{\partial t} = -\frac{x}{t^2}\mathcal{H}_t(x) + \frac{x}{t}\frac{\partial^2\mathcal{H}_t(x)}{\partial x^2}$$

$$\frac{\partial u}{\partial x^2} = \frac{2}{t}\frac{\partial \mathcal{H}_t(x)}{\partial x} + \frac{x}{t}\frac{\partial^2\mathcal{H}_t(x)}{\partial x^2}.$$

Note that $$\frac{2}{t}\frac{\partial\mathcal{H}_t(x)}{\partial x} = \frac{2}{t}\frac{-x}{2t}\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t} = -\frac{x}{t^2}\mathcal{H}_t(x)$$

This shows that $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

$\square$

(b) ...

(c) $u\left(x, \frac{x^2}{4c}\right)$ $\to \infty$ as $x\to 0$

$\square$