The Gradient Flow for Gauged Harmonic Map in Dimension Two I

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ABSTRACT: In this article, we study a gradient flow associated with a gauged harmonic map energy in dimension two. Some specific properties are considered, for instances bubbling analysis, asymptotic behavior and removability of singularities.

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I. Introduction

The theories of harmonic map and Yang-Mills play fundamental roles in the study of both physics and geometry. In this work, we couple these two theories and study a model of gauged harmonic map. Our motivations stem from the work of [1], [2], [11], [16] and the references therein. But as one will see, we define our model in more general settings, which involve a non-Abelian structure group $\mathscr{G}$ and a fibre bundle $\mathscr{E}$, whose typical fibre $\mathscr{N}$ is a closed $\mathscr{G}$-invariant Riemannian manifold.

I.1. Gauged Harmonic Map and Gradient Flow

Let $\mathscr{M}$ and $\mathscr{N}$ be two closed Riemannian manifolds of dimensions m and n, respectively. We assume that $\mathscr{M}$ is equipped with a Riemannian metric g and $\mathscr{N}$ is isometrically embedded into $\mathbb{R}^L$. Suppose that

$$\mathscr{G} \subseteq SO(L)$$

is a compact Lie group with Lie algebra $\mathbf{g}$. Naturally, if we identify an element in $\mathscr{N}$ as a column vector in $\mathbb{R}^L$ through the isometrical embedding, then $\mathscr{G}$ induces a smooth left action on $\mathscr{N}$ by the left multiplication of matrix. In this article, we require that $\mathscr{N}$ is $\mathscr{G}$-invariant. That is

$$g \cdot \mathscr{N} \subseteq \mathscr{N}, \qquad \forall\ g \in \mathscr{G}.$$

In order to introduce the model of gauged harmonic map, we need some geometric terminology. Let $\{\mathscr{U}_\alpha\}$ be a finite open covering of $\mathscr{M}$. On $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$, we define a smooth map

$$g_{\alpha,\beta} : \mathscr{U}_\alpha \cap \mathscr{U}_\beta \longmapsto \mathscr{G}.$$

Obviously, if $\{g_{\alpha,\beta}\}$ satisfies the co-cycle condition, then it determines a principal $\mathscr{G}$-bundle, denoted by $\mathscr{P}$, over $\mathscr{M}$. By $\mathscr{P}$, we can construct a fibre bundle $\mathscr{E} = \mathscr{P} \times_{\mathscr{G}} \mathscr{N}$ via the left action of $\mathscr{G}$ on $\mathscr{N}$ which was discussed above. It is clear that $\mathscr{E}$ is a sub-bundle of $\mathscr{F} = \mathscr{P} \times_{\mathscr{G}} \mathbb{R}^L$.

The variables in the theory of gauged harmonic map are a connection 1-form A on $\mathscr{P}$ and a section ϕ of $\mathscr{E}$. Locally on $\mathscr{U}_\alpha$, A and ϕ can be represented as

$$A_\alpha : \mathscr{U}_\alpha \longmapsto \mathbf{g} \qquad \text{and} \qquad \phi_\alpha : \mathscr{U}_\alpha \longmapsto \mathscr{N},$$

respectively. Moreover, on $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$,

$$A_\alpha = \mathrm{Ad}_{g_{\alpha,\beta}}(A_\beta) - \mathrm{d}g_{\alpha,\beta} \cdot g_{\beta,\alpha} \qquad \text{and} \qquad \phi_\alpha = g_{\alpha,\beta}\phi_\beta,$$

where Ad is the adjoint representation of $\mathscr{G}$. The energy functional associated with (A, ϕ) is defined by

$$E(A,\phi) = \int_{\mathscr{M}} e(A,\phi)\, \mathrm{d}v_g, \tag{1.1}$$

where

$$e(A,\phi) = \frac{1}{2}\left(|F_A|^2 + |D_A\phi|^2\right)$$

is the energy density. Note that in the definition of $e(A,\phi)$, F_A is the curvature 2-form, while $D_A\phi$ is the covariant derivative induced by A. The norms in $e(A,\phi)$ are defined in a natural way, using the metrics on $\mathscr{M}$, $\mathbb{R}^L$ and the Killing form of the Lie algebra $\mathbf{g}$. With the energy functional $E(A,\phi)$, we define gauged harmonic map to be a critical point of (1.1). More precisely,

Definition 1.1 (Gauged Harmonic Map). *A section $\phi \in \Omega^0(\mathscr{E})$ is called gauged harmonic map if there is a connection A such that (A, ϕ) satisfies the following Euler-Lagrange equation of (1.1):*

$$\begin{cases} D_A^* F_A = -\left(g_l \phi, D_A \phi\right) g_l; \\ \\ D_A^* D_A \phi = \left(D_A \nu_i(\phi), D_A \phi\right) \nu_i(\phi). \end{cases} \tag{1.2}$$

D_A^* is the formal adjoint operator of D_A. $\{g_l\}$ $(l = 1, ..., k)$ is an orthonormal basis of the Lie algebra $\mathbf{g}$ under the inner product $\langle \cdot, \cdot \rangle$ which is induced from the Killing form on $so(L)$.

$$\{\nu_i(\phi)\}, \qquad i = 1, ..., L - n$$

is an orthonormal frame of the normal bundle $(T\mathscr{N})^\perp$ at ϕ. Similarly as in the work of harmonic map and Yang-Mills, we can introduce a gradient flow associated with the energy functional (1.1) as follows:

$$\begin{cases} \partial_t A = -D_A^* F_A - \left(g_l \phi, D_A \phi\right) g_l; \\ \\ \partial_t \phi = -D_A^* D_A \phi + \left(D_A \nu_i(\phi), D_A \phi\right) \nu_i(\phi). \end{cases} \tag{1.3}$$

Note that (1.2) is gauge invariant under the gauge transformation

$$s \cdot (A, \phi) = (s \cdot A, s \cdot \phi),$$

where $s \cdot A = \mathrm{Ad}_s(A) - \mathrm{d}s \cdot s^{-1}$, $s \cdot \phi = s\phi$, while (1.3) is also gauge invariant under a time-independent gauge transformation s.

I.2. Main Results

There are two directions in the study of gauged harmonic map. The first one is to reduce the problem to a first-order Bogomol'nyi type equation (vortex equation) by studying the lowest bound of energy functional in a homotopy class. For instances, [1], [2], [11], [16] and the references therein. To solve the vortex equation, particularly in Abelian case, one can apply either the method of Taubes (see [11]) or a stability criterion based on Hitchin-Kobayashi correspondence (see [2]). In fact, Taubes' method works pretty well when the base manifold $\mathscr{M}$ is a Riemannian surface or $\mathbb{C}$, while the stability method has its application in the case when $\mathscr{M}$ is a Kähler manifold with higher complex dimension. Our approach follows the second direction. That is to study the gradient flow associated with the energy functional (1.1). Along this direction, many works have been carried out in the theory of harmonic map (see [10], [18], [20]), Yang-Mills (see [13], [15], [17], [22]) and Yang-Mills-Higgs (see [5]-[7]). As is well-known, the critical dimension for the heat flow of harmonic map is 2, while the critical dimension for the Yang-Mills flow is 4. Therefore, when $\dim(\mathscr{M}) = 2$, our model is subcritical for the Yang-Mills fields and critical for the section of $\mathscr{E}$.

We now describe the organization of this article. In Section II, we prove the local existence of the gradient flow (1.3) with smooth initial data (A_0, ϕ_0). More precisely, we show that

Theorem 1.2. *There is a $T > 0$ so that the gradient flow (1.3) admits a smooth solution on $[0, T)$ with the given smooth initial data (A_0, ϕ_0). For $p > \dim(\mathscr{M})$, T can be shown to depend on the $W^{2,p}$-norm of (A_0, ϕ_0).*

Theorem 1.2 works for any dimension. Start from Section III. We assume that $\dim(\mathscr{M}) = 2$ and study some specific properties associated with the gradient flow (1.3), for instances bubbling analysis, asymptotic behavior and removability of singularities. Section III is a preparation, in which we show local energy inequalities, Bochner-type inequality and ϵ-regularity. A criterion is given in Section III.4 for the first singular time T_0 of the gradient flow (1.3). If $T_0 < \infty$, then the bubbling phenomenon occurs at T_0. In this case, we show that

Theorem 1.3. *Suppose that* $\dim(\mathscr{M}) = 2$*. If the first singular time* T_0 *is finite, then there exist a set of finitely many points in* $\mathscr{M}$*, denoted by* $\{x_i\}$*, so that for all* $k \in \mathbb{N}$*,*

$$(A(t), \phi(t)) \longrightarrow (A(T_0), \phi(T_0)), \qquad \text{in } C^k_{\text{loc}}(\mathscr{M} \setminus \{x_i\}), \quad \text{as } t \uparrow T_0.$$

Moreover, there exist finitely many non-trivial harmonic maps from $\mathbb{R}^2$ *into* $\mathscr{N}$*, denoted by* $\{\phi_s^*\}$*, so that the following energy identity holds:*

$$\lim_{t\uparrow T_0} \int_{\mathscr{M}} e(t)\,\mathrm{d}v_g = \int_{\mathscr{M}} e(T_0)\,\mathrm{d}v_g + \frac{1}{2}\sum_s \int_{\mathbb{R}^2} |\nabla \phi_s^*|^2\,\mathrm{d}x. \tag{1.4}$$

Conventionally, $\{\phi_s^*\}$ in Theorem 1.3 are called bubbles. From (1.4), one realizes that due to the existence of singluar points, the gradient flow (1.3) loses some energy at T_0. Moreover, the lost energy can be recovered by finitely many harmonic maps from $\mathbb{R}^2$ into $\mathscr{N}$. Different from the assumptions in Theorem 1.3, in Section V, we suppose that the gradient flow (1.3) admits a global smooth solution on $[0, \infty)$. We are interested in the asymptotic behavior of the global solution as $t \uparrow \infty$. In fact, we have

Theorem 1.4. *Suppose that* (A, ϕ) *is a global smooth solution of (1.3). Then there exist* $t_k \uparrow \infty$ *and a finite covering* $\{B_i^*\}$ *of* $\mathscr{M}$ *so that the followings hold:*

(1). For each k *and* i*,* $(A(t_k), \phi(t_k))$ *is gauge equivalent to some smooth* $(A^*_{k,i}, \phi^*_{k,i})$ *on* B_i^**. Define*

$$A_k^*|_{B_i^*} = A^*_{k,i}, \qquad \phi_k^*|_{B_i^*} = \phi^*_{k,i}, \qquad \text{for all } i.$$

Then there exists a principal $\mathscr{G}$*-bundle* $\mathscr{P}_k$ *over* $\mathscr{M}$ *so that* A_k^* *is a smooth connection on* $\mathscr{P}_k$ *and* ϕ_k^* *is a smooth section on the associated fibre bundle* $\mathscr{E}_k = \mathscr{P}_k \times_{\mathscr{G}} \mathscr{N}$*;*

(2). As $k \to \infty$*, we have a smooth principal* $\mathscr{G}$*-bundle* $\mathscr{P}^*$ *over* $\mathscr{M}$ *so that*

$$\mathscr{P}_k \longrightarrow \mathscr{P}^*, \qquad \mathscr{E}_k \longrightarrow \mathscr{E}^* = \mathscr{P}^* \times_{\mathscr{G}} \mathscr{N}.$$

Here, the convergence of principal $\mathscr{G}$*-bundles and fibre bundles are defined to be the* C^∞*-convergence of the associated transition functions;*

(3). There are a $W^{1,2}$*-connection* A_∞ *on* $\mathscr{P}^*$ *and a* $W^{1,2}$*-section* ϕ_∞ *on* $\mathscr{E}^*$ *so that* A_∞ *and* ϕ_∞ *are smooth away from points in* Σ*, where* Σ *is a finite subset of* $\mathscr{M}$*. Moreover,*

$$(A_k^*, \phi_k^*) \longrightarrow (A_\infty, \phi_\infty), \qquad \text{in } C^\infty_{\text{loc}}(\mathscr{M} \setminus \Sigma), \quad \text{as } k \to \infty;$$

(4). (A_∞, ϕ_∞) *solves (1.2) smoothly away from the points in* Σ*.*

Similarly as in the case of harmonic maps (see [14]), we can remove the singularities in Σ from (A_∞, ϕ_∞) and show in Section VI that

Theorem 1.5 (Removability of Singularities). (A_∞, ϕ_∞) *is a global smooth solution of (1.2) on* $\mathscr{M}$*.*

II. Local Existence

Let A_0 be a smooth connection 1-form on $\mathscr{P}$ and $\phi_0 \in \Omega^0(\mathscr{E})$ be a smooth section. Here in the following, we study the local existence of the gradient flow (1.3) with the given initial data (A_0, ϕ_0). One should refer to [21] for some standard terminology in the gauge field theory. Sobolev spaces of sections of vector bundles are also introduced in [21].

Now we sketch the plan of this section. Note that the first equation in (1.3) is just partially parabolic. Therefore, in Section II.1 below, we use gauge transformation, similarly as the work of Donaldson, to reduce the gradient flow (1.3) into a parabolic gauge equivalent flow. Furthermore, we use the projection near the manifold $\mathscr{N} \hookrightarrow \mathbb{R}^L$ to get rid of the constraint on the range of unknown section. In such a way, we obtain an extended gauge equivalent flow. The linear theory of parabolic equation on vector bundles and contraction mapping theorem then can be applied to attain a unique smooth solution of the extended gauge equivalent flow. We then show, similarly as the heat flow of harmonic map (see [10]) and liquid crystal flow (see [9] and [20]), that if the initial section lies in $\Omega^0(\mathscr{E})$, then the solution of the extended gauge equivalent flow must be a solution of the parabolic gauge equivalent flow, which, furthermore, implies a solution of the original gradient flow (1.3).

II.1. Gauge Equivalent Flow and Its Extension

We reduce the gradient flow (1.3) into a parabolic gauge equivalent flow. Suppose that S is a gauge transformation and $(\bar{A}, \psi)$ is gauge equivalent to (A, ϕ) via S^{-1}. That is

$$\bar{A} = S^{-1} \cdot A, \qquad \psi = S^{-1} \cdot \phi.$$

It is clear that if (A, ϕ) satisfies (1.3), then $(\bar{A}, \psi)$ solves

$$\begin{cases} \partial_t \bar{A} = -D^*_{\bar{A}} F_{\bar{A}} - (g_l \psi, D_{\bar{A}} \psi)\, g_l + D_{\bar{A}} \left(S^{-1} \cdot \partial_t S\right); \\ \partial_t \psi = -D^*_{\bar{A}} D_{\bar{A}} \psi + (D_{\bar{A}} \nu_i(\psi), D_{\bar{A}} \psi)\, \nu_i(\psi) - \left(S^{-1} \cdot \partial_t S\right) \cdot \psi. \end{cases} \tag{2.1}$$

Let A_{ref} be a smooth reference connection and suppose that $\bar{A} = A_{\mathrm{ref}} + a$. By requiring that

$$S^{-1} \cdot \partial_t S = -D^*_{\mathrm{ref}} a, \tag{2.2}$$

one may then rewrite the equation (2.1) in terms of (a, ψ) as follows:

$$\begin{cases} \partial_t a + \Delta_{\mathrm{ref}} a = f(a, \psi) - (g_l \psi, D_{\mathrm{ref}} \psi) g_l - D^*_{\mathrm{ref}} F_{\mathrm{ref}}; \\ \partial_t \psi + \nabla^*_{\mathrm{ref}} \nabla_{\mathrm{ref}} \psi = (D_{\bar{A}} \nu_i(\psi), D_{\bar{A}} \psi)\, \nu_i(\psi) + 2a^k \nabla_{\mathrm{ref},k} \psi + a^k a_k \psi, \end{cases} \tag{2.3}$$

where F_{ref} is the curvature 2-form of A_{ref}, ∇_{ref} is the induced covariant derivative and

$$f(a, \psi) = a \times F_{\mathrm{ref}} + a \times \nabla_{\mathrm{ref}} a - (g_l \psi, a\psi) g_l + a \times a + a \times a \times a.$$

In the above, $\times$ denotes any multi-linear map with smooth coefficients.

$$\Delta_{\mathrm{ref}} = D^*_{\mathrm{ref}} D_{\mathrm{ref}} + D_{\mathrm{ref}} D^*_{\mathrm{ref}}$$

is the Hodge Laplacian. System (2.3) is called parabolic gauge equivalent flow corresponding to (1.3).

Note that the unknown variable ψ in (2.3) can be represented locally as a map into $\mathscr{N}$. To get rid of the constraint on the range of ψ, we need a smooth projection

$$\Pi : \mathscr{N}_{3\delta} \longmapsto \mathscr{N}, \qquad \text{for some } \delta > 0.$$

Here $\mathscr{N}_{3\delta}$ is the 3δ-neighborhood of $\mathscr{N}$ in $\mathbb{R}^L$. Let ρ_1 be a smooth non-negative function such that

$$\rho_1(s) = \begin{cases} 1, & \text{if } s \in [0, \delta]; \\ \leq 1, & \text{if } s \in [\delta, 2\delta]; \\ 0, & \text{if } s \geq 2\delta. \end{cases}$$

ρ_2 is a cut-off function, which is defined by

$$\rho_2(x) = \rho_1\left(\text{dist}\,(x, \mathscr{N})\right), \qquad \forall\, x \in \mathbb{R}^L.$$

Obviously, ρ_2 is $\mathscr{G}$-invariant. That is $\forall\ x \in \mathbb{R}^L$, $g \in \mathscr{G}$, we have

$$\rho_2(g \cdot x) = \rho_2(x).$$

By the projection Π and the cut-off function ρ_2, one can define an extended gauge equivalent flow as follows:

$$\begin{cases} \partial_t a + \Delta_{\text{ref}} a = f(a, \psi) - (g_l \psi, D_{\text{ref}} \psi) g_l - D^*_{\text{ref}} F_{\text{ref}}; \\ \partial_t \psi + \nabla^*_{\text{ref}} \nabla_{\text{ref}} \psi = \rho_2(\psi) \left((D_{\bar{A}} \nu_i(\psi), D_{\bar{A}} (\Pi\psi))\, \nu_i(\psi) + 2a^k \nabla_{\text{ref},k} \Pi\psi + a^k a_k \Pi\psi \right). \end{cases} \tag{2.4}$$

Here ψ is an unkown section on $\mathscr{F}$. $\nu_i(\psi)$ should be understood as the i-th normal direction at $\Pi\psi$.

II.2. Estimates for linear heat equation on vector bundles

Denote by Q_T the cylinder $\mathscr{M} \times [0, T]$. With the given $f \in L^p(Q_T)$ and $\phi^0 \in W^{2,p}(\mathscr{M}; \mathscr{F})$, we study the linear parabolic system defined as follows:

$$\begin{cases} \partial_t \phi + \nabla^*_{\text{ref}} \nabla_{\text{ref}} \phi = f; \\ \phi|_{t=0} = \phi^0. \end{cases} \tag{2.5}$$

Basically, there are two estimates important to us. The first one is the $W^{2,1}_p$ - estimate for the solution ϕ of (2.5). Another one is the $L^\infty W^{1,p}$-estimate. We consider these two estimates in Proposition 2.1 and 2.2, respectively. In the following, $p > 2$ is a fixed constant.

Proposition 2.1. *The system (2.5) admits a unique solution such that*

$$\|\phi\|_{W^{2,1}_p(Q_T)} \lesssim \|f\|_{L^p(Q_T)} + \left\|\phi^0\right\|_{W^{2,p}}.$$

Proof. Firstly, we reduce the system (2.5) into the case in which $\phi^0 \equiv 0$. Let $\Sigma = \{\mathscr{U}_\alpha\}$ be the finite open covering of $\mathscr{M}$, by which the principal bundle $\mathscr{P}$ is defined. Suppose that

$$\rho_\alpha \in C^\infty_c(\mathscr{U}_\alpha)$$

is a sequence of non-negative functions subordinate to the covering Σ. We require that

$$\sum_\alpha \rho_\alpha^2 \equiv 1, \qquad \text{in } \mathscr{M}.$$

Fix an α. $\rho_\alpha \phi^0_\alpha$ is in fact a map from $\mathbb{R}^m$ to $\mathbb{R}^L$. Here ϕ^0_α is the local representation of ϕ^0 in $\mathscr{U}_\alpha$. Define

$$\psi_\alpha = \Gamma_t * \left(\rho_\alpha \phi^0_\alpha\right),$$

where $\Gamma_t(x) = \Gamma(x,t)$ is the standard heat kernel on $\mathbb{R}^m$ and $*$ is the spatial convolution operator on $\mathbb{R}^m$. One can easily check that

$$\operatorname*{ess\,sup}_{t>0} \|\psi_\alpha\|^p_{W^{2,p}} \lesssim \left\|\phi^0_\alpha\right\|^p_{W^{2,p}(\mathscr{U}_\alpha)}.$$

Denote by $\bar{\psi}_\alpha$ the restriction of ψ_α on $\mathscr{U}_\alpha$ and patch them together by setting

$$\Psi_\alpha(\cdot,t) = \sum_\gamma \rho_\gamma\, g_{\alpha,\gamma}\, \bar{\psi}_\gamma(\cdot,t), \qquad \text{in } \mathscr{U}_\alpha.$$

We claim that

1. On $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$, $\Psi_\alpha = g_{\alpha,\beta}\Psi_\beta$. Hence, $\Psi(\cdot,t) \in W^{2,p}\left(\mathscr{M};\mathscr{F}\right)$ by our construction;

2. At $t=0$,

$$\Psi_\alpha(\cdot,0) = \sum_\gamma \rho_\gamma\, g_{\alpha,\gamma}\, \bar{\psi}_\gamma(\cdot,0) = \sum_\gamma \rho^2_\gamma\, g_{\alpha,\gamma}\, \phi^0_\gamma = \sum_\gamma \rho^2_\gamma\, \phi^0_\alpha = \phi^0_\alpha.$$

Therefore, $\Psi(\cdot,0) = \phi^0$;

3. Ψ admits a $W^{2,1}_p$-estimate shown as follows:

$$\operatorname*{ess\,sup}_{t>0} \left(\int_{\mathscr{M}} |\partial_t \Psi|^p \ \mathrm{d}v_g + \|\Psi\|^p_{W^{2,p}} \right) \lesssim \left\|\phi^0\right\|^p_{W^{2,p}}. \tag{2.6}$$

The estimate on $\partial_t\Psi$ in (2.6) can be derived by noticing that

$$\partial_t \psi_\alpha = \Delta \psi_\alpha, \qquad \text{in } \mathbb{R}^m.$$

Here Δ is the standard Laplace operator in $\mathbb{R}^m$.

We define $\Phi = \phi - \Psi$. It is clear that Φ satisfies

$$\begin{cases} \partial_t \Phi + \Delta_{\mathrm{ref}} \Phi = \hat{f} := f - \left(\partial_t \Psi + \Delta_{\mathrm{ref}} \Psi\right); \\ \Phi|_{t=0} = 0. \end{cases} \tag{2.7}$$

Notice (2.6), we know that $\hat{f} \in L^p(Q_T)$. Now we apply Proposition 2.7 in [22] and imply that

$$\|\Phi\|_{W^{2,1}_p(Q_T)} \lesssim \|\hat{f}\|_{L^p(Q_T)} \lesssim \|f\|_{L^p(Q_T)} + \left\|\phi^0\right\|_{W^{2,p}}. \tag{2.8}$$

Combine the above inequality with (2.6), we have

$$\|\phi\|_{W^{2,1}_p(Q_T)} \lesssim \|f\|_{L^p(Q_T)} + \left\|\phi^0\right\|_{W^{2,p}}.$$

The proof is then finished. □

As for the $L^\infty W^{1,p}$-estimate for the solution of (2.5), we have

Proposition 2.2. *Let ϕ be the unique solution of (2.5). Then*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\phi\|^p_{W^{1,p}} \lesssim \|f\|^p_{L^p(Q_T)} + \left\|\phi^0\right\|^p_{W^{2,p}}.$$

Proof. Act ∇_{ref} on both sides of (2.7) and inner product with $p\,|\nabla_{\text{ref}}\Phi|^{p-2}\,\nabla_{\text{ref}}\Phi$. One has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}}|\nabla_{\text{ref}}\Phi|^{p}+p\int_{\mathscr{M}}\left(|\nabla_{\text{ref}}\Phi|^{p-2}\,\nabla_{\text{ref}}\Phi,\nabla_{\text{ref}}\Delta_{\text{ref}}\Phi\right)=p\int_{\mathscr{M}}\left(|\nabla_{\text{ref}}\Phi|^{p-2}\,\nabla_{\text{ref}}\Phi,\nabla_{\text{ref}}\hat{f}\right).$$

In the above integral and the integral in the following, we omit $\mathrm{d}v_g$ for convenience. Integrate by parts for the right-hand side and the second term on the left-hand side above. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}}|\nabla_{\text{ref}}\Phi|^{p}+p\int_{\mathscr{M}}|\nabla_{\text{ref}}\Phi|^{p-2}\,|\Delta_{\text{ref}}\Phi|^{2}-p\int_{\mathscr{M}}|\nabla_{\text{ref}}\Phi|^{p-2}\left(\Delta_{\text{ref}}\Phi,\hat{f}\right)=$$

$$=p\,(p-2)\int_{\mathscr{M}}|\nabla_{\text{ref}}\Phi|^{p-4}\left(\Delta_{\text{ref}}\Phi-\hat{f},\nabla^{2}_{\text{ref}}\Phi\left(\nabla_{\text{ref}}\Phi,\nabla_{\text{ref}}\Phi\right)+F_{\text{ref}}\Phi\left(\nabla_{\text{ref}}\Phi,\nabla_{\text{ref}}\Phi\right)\right).$$

Fix an arbitrary $\tau\in[0,T]$ and integrate the above equality with respect to t from 0 to τ. One may imply by (2.8) and Hölder's inequality that

$$\operatorname*{ess\,sup}_{\tau\in[0,T]}\int_{\mathscr{M}}|\nabla_{\text{ref}}\Phi|^{p}\lesssim\|f\|^{p}_{L^{p}(Q_T)}+\left\|\phi^{0}\right\|^{p}_{W^{2,p}}.$$

Notice (2.6). The proof is then finished. ☐

Simlar arguments can be applied to 1-forms. In fact, we have

Proposition 2.3. *Suppose that $f\in L^{p}\left([0,T];L^{p}(T^{*}\mathscr{M}\otimes\mathrm{Ad}\mathscr{P})\right)$ and $a^{0}\in W^{2,p}(T^{*}\mathscr{M}\otimes\mathrm{Ad}\mathscr{P})$. If a is the unique solution of the system:*

$$\begin{cases}\partial_{t}a+\Delta_{\text{ref}}a=f;\\[2mm] a\big|_{t=0}=a^{0},\end{cases}\tag{2.9}$$

where Δ_{ref} is the Hodge Laplacian, then one has

$$\|a\|_{W^{2,1}_{p}(Q_T)}+\operatorname*{ess\,sup}_{t\in[0,T]}\|a\|_{W^{1,p}}\lesssim\|f\|_{L^{p}(Q_T)}+\left\|a^{0}\right\|_{W^{2,p}}.$$

II.3. Local Existence for the Gradient Flow

In this section, we assume that $p>m$.

$$a^{0}\in\Omega^{1}(\mathrm{Ad}\mathscr{P})\qquad\text{and}\qquad\psi^{0}\in\Omega^{0}(\mathscr{F})$$

are initial datum corresponding to the extended gauge equivalent flow (2.4). Without loss of generality, we choose $T<1$ and suppose that $V^{g}_{p,T}$ and $V^{s}_{p,T}$ are closures of

$$C^{\infty}_{0+}\left([0,T];\Omega^{1}(\mathrm{Ad}\mathscr{P})\right)\qquad\text{and}\qquad C^{\infty}_{0+}\left([0,T];\Omega^{0}(\mathscr{F})\right)$$

under the norms

$$\|\cdot\|^{p}_{V^{g}_{p,T}}=\operatorname*{ess\,sup}_{t\in[0,T]}\|\cdot\|^{p}_{W^{1,p}(T^{*}\mathscr{M}\otimes\mathrm{Ad}\mathscr{P})}+\int_{0}^{T}\int_{\mathscr{M}}|\nabla^{2}_{\text{ref}}\ \cdot|^{p}$$

and

$$\| \cdot \|^p_{V^s_{p,T}} = \operatorname*{ess\,sup}_{t\in[0,T]} \| \cdot \|^p_{W^{1,p}(\mathscr{M};\mathscr{F})} + \int_0^T \int_{\mathscr{M}} \left|\nabla^2_{\mathrm{ref}} \ \cdot\right|^p ,$$

respectively. Here, all smooth 1-forms in $C^\infty_{0+}\left([0,T];\Omega^1(\mathrm{Ad}\mathscr{P})\right)$ and sections in $C^\infty_{0+}\left([0,T];\Omega^0(\mathscr{F})\right)$ take zero initial values. By $V^g_{p,T}$ and $V^s_{p,T}$, we define $V_{p,T} := V^g_{p,T} \times V^s_{p,T}$, which is equipped with the norm

$$\| \cdot \|_{V_{p,T}} = \| \cdot \|_{V^g_{p,T}} + \| \cdot \|_{V^s_{p,T}}.$$

Notice that, by Sobolev embedding,

$$\|f\|_{L^\infty(Q_T)} \lesssim \|f\|_{V_{p,T}}, \qquad \forall\, f \in V_{p,T}. \tag{2.10}$$

Proposition 2.4. *With the given smooth initial datum $\left(a^0,\psi^0\right)$, there exists a $T>0$ such that the extended gauge equivalent flow (2.4) admits a unique smooth solution in $[0,T)$. T depends on the p and $W^{2,p}$-norm of (a^0,ψ^0).*

Proof. Let $f\equiv 0$. We solve the homogeneous equation of (2.9) and (2.5) with the given initial datum a^0 and ψ^0. The solutions are denoted by a^1 and ψ^1, respectively. By Proposition 2.1-2.3, we have

$$\left\|\left(a^1,\psi^1\right)\right\|_{V_{p,T}} \lesssim \left\|\left(a^0,\psi^0\right)\right\|_{W^{2,p}}. \tag{2.11}$$

Decompose the unknow variable (a,ψ) as $a=a^1+\bar a$, $\psi=\psi^1+\bar\psi$. Therefore, one can rewrite the equation (2.4) in terms of $\left(\bar a,\bar\psi\right)$ as follows:

$$\begin{cases} \partial_t \bar a + \Delta_{\mathrm{ref}} \bar a = f_1\left(\bar a,\bar\psi\right); \\ \partial_t \bar\psi + \nabla^*_{\mathrm{ref}}\nabla_{\mathrm{ref}}\bar\psi = g_1\left(\bar a,\bar\psi\right), \end{cases} \tag{2.12}$$

where $f_1\left(\bar a,\bar\psi\right)$ and $g_1\left(\bar a,\bar\psi\right)$ are defined to be

$$f\left(\bar a + a^1, \bar\psi+\psi^1\right) - \left(g_l\left(\bar\psi+\psi^1\right), D_{\mathrm{ref}}\bar\psi + D_{\mathrm{ref}}\psi^1\right) g_l - D^*_{\mathrm{ref}}F_{\mathrm{ref}}$$

and $g\left(\bar a + a^1, \bar\psi+\psi^1\right)$, respectively. Here $f(a,\psi)$ is defined in (2.3) and $g(a,\psi)$ stands for the right-hand side of the second equation in (2.4).

We use the contraction mapping theorem to solve (2.12) with 0 initial datum. In the following, C is a suitably large constant depending on p, $\mathscr{M}$, $\mathscr{N}$, D_{ref} and the $W^{2,p}$-norms of a^0 and ψ^0. Fix $\left(\bar a_*,\bar\psi_*\right)\in B_{r_0,T}$, where $r_0<1$ and $B_{r_0,T}$ is the ball in $V_{p,T}$ with center 0 and radius r_0. We consider the system

$$\begin{cases} \partial_t \bar a + \Delta_{\mathrm{ref}} \bar a = f_1\left(\bar a_*,\bar\psi_*\right); \\ \partial_t \bar\psi + \nabla^*_{\mathrm{ref}}\nabla_{\mathrm{ref}}\bar\psi = g_1\left(\bar a_*,\bar\psi_*\right), \end{cases} \tag{2.13}$$

with 0 initial value. By (2.10)-(2.11), we know that

$$f_1\left(\bar a_*,\bar\psi_*\right) \in L^p\left([0,T];L^p\left(T^*\mathscr{M}\otimes \mathrm{Ad}\mathscr{P}\right)\right)$$

and can be estimated by

$$\int_{Q_T} \left|f_1\left(\bar a_*,\bar\psi_*\right)\right|^p \le C\,T.$$

Apply Proposition 2.3, there exists a unique solution $\bar{a}$ of the first equation in (2.13) and moreover,

$$\|\bar{a}\|^p_{V^g_{p,T}} \le C\,T. \tag{2.14}$$

Similar arguments can be applied to $\bar{\psi}$ which is the solution for the second equation in (2.13). Note that

$$\int_{Q_T} \left|g_1(\bar{a}_*, \bar{\psi}_*)\right|^p \le C\ \left(T + \int_{Q_T} \left|\nabla_{\mathrm{ref}}\bar{\psi}_*\right|^{2p} + \left|\nabla_{\mathrm{ref}}\psi^1\right|^{2p}\right). \tag{2.15}$$

We estimate $\int_{Q_T} \left|\nabla_{\mathrm{ref}}\bar{\psi}_*\right|^{2p}$ in (2.15). In one way, it can be bounded by

$$\int_0^T \|\nabla_{\mathrm{ref}}\bar{\psi}_*\|^p_{L^\infty(\mathscr{M})} \int_{\mathscr{M}} \left|\nabla_{\mathrm{ref}}\bar{\psi}_*\right|^p \le \operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathscr{M}} \left|\nabla_{\mathrm{ref}}\bar{\psi}_*\right|^p \cdot \int_0^T \|\nabla_{\mathrm{ref}}\bar{\psi}_*\|^p_{L^\infty(\mathscr{M})}.$$

In another way, by Sobolev embedding, one may estimate the last term above by

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathscr{M}} \left|\nabla_{\mathrm{ref}}\bar{\psi}_*\right|^p \cdot \int_0^T \|\nabla_{\mathrm{ref}}\bar{\psi}_*\|^p_{W^{1,p}} \le C\ \|\bar{\psi}_*\|^{2p}_{V^s_{p,T}}.$$

Since $\bar{\psi}_* \in B_{r_0,T}$, we know that

$$\int_{Q_T} \left|\nabla_{\mathrm{ref}}\bar{\psi}_*\right|^{2p} \le C\ \|\bar{\psi}_*\|^{2p}_{V^s_{p,T}} \le C\ r_0^{2p}.$$

Similarly, for $\int_{Q_T} |\nabla_{\mathrm{ref}}\psi^1|^{2p}$, we have

$$\int_{Q_T} \left|\nabla_{\mathrm{ref}}\psi^1\right|^{2p} \le C\ \operatorname*{ess\,sup}_{0\le t\le T} \int_{\mathscr{M}} \left|\nabla_{\mathrm{ref}}\psi^1\right|^p \cdot \int_{Q_T} \left|\nabla_{\mathrm{ref}}\psi^1\right|^p + \left|\nabla^2_{\mathrm{ref}}\psi^1\right|^p \le C\left(T + \int_{Q_T} \left|\nabla^2_{\mathrm{ref}}\psi^1\right|^p\right).$$

Therefore, one can estimate $g_1(\bar{a}_*, \bar{\psi}_*)$ as follows:

$$\int_{Q_T} \left|g_1\left(\bar{a}_*, \bar{\psi}_*\right)\right|^p \le C\ \left(T + r_0^{2p} + \int_{Q_T} \left|\nabla^2_{\mathrm{ref}}\psi^1\right|^p\right). \tag{2.16}$$

Then by Proposition 2.1 and 2.2, we have

$$\|\bar{\psi}\|^p_{V^s_{p,T}} \le C\ \left(T + r_0^{2p} + \int_{Q_T} \left|\nabla^2_{\mathrm{ref}}\psi^1\right|^p\right). \tag{2.17}$$

By (2.14) and (2.17), we know that if T and r_0 are suitably small, the solution $\left(\bar{a}, \bar{\psi}\right)$ of (2.13) lies in $B_{r_0,T}$. Here we used the absolute continuity of $\int_{Q_T} \left|\nabla^2_{\mathrm{ref}}\psi^1\right|^p$. Now we can construct a nonlinear operator which sends $(\bar{a}_*, \bar{\psi}_*) \in B_{r_0,T}$ to the unique solution of (2.13). Clearly, this nonlinear operator is also a contraction mapping between $B_{r_0,T}$ and itself when r_0 and T are suitably small. The local existence for the extended gauge equivalent flow is then obtained. The smoothness of the solution can be easily obtained by standard parabolic estimates. We omit the arguments here. □

In the following, we show that the smooth solution for the extended gauge equivalent flow (2.4) is also a solution for the parabolic gauge equivalent flow (2.3) if the initial section is a section of the fibre bundle $\mathscr{E}$. In fact, we have

Proposition 2.5. *With given initial data $(a^0, \phi^0) \in \Omega^1(\mathrm{Ad}\mathscr{P}) \times \Omega^0(\mathscr{E})$, there exists a $T > 0$ such that the parabolic gauge equivalent flow (2.3) admits a unique smooth solution in $[0, T)$, where T depends on p and the $W^{2,p}$-norm of (a^0, ϕ^0).*

Proof. Suppose that (a,ψ) is the unique solution of (2.4) with the given initial datum (a^0,ϕ^0). By the regularity of the extended gauge equivalent flow (2.4), we know that when T is small enough, ψ takes its value in the δ-neighborhood of $\mathscr{N}$. Therefore, $\rho_2(\psi)\equiv 1$ in $[0,T)$ and the second equation in (2.4) can be read as

$$\partial_t\psi+\nabla^*_{\mathrm{ref}}\nabla_{\mathrm{ref}}\psi=\left(D_{\bar{A}}\nu_i(\psi),D_{\bar{A}}\Pi\psi\right)\nu_i(\psi)+2a^k\nabla_{\mathrm{ref},k}\Pi\psi+a^ka_k\Pi\psi. \tag{2.18}$$

Here $\bar{A}=A_{\mathrm{ref}}+a$. Define $\rho=\frac{1}{2}|\psi-\Pi\psi|^2$. Then by Lemma 2.6 below, we know that

$$\partial_t\rho+\nabla^*\nabla\rho\le 0.$$

Here $-\nabla^*\nabla=\Delta_{\mathscr{M}}$ is the Laplace-Beltrami operator on the mainifold $\mathscr{M}$. The standard maximum principle implies that $\rho\equiv 0$ in $[0,T)$. The proof is then finished. □

We complete the proof of Proposition 2.5 by showing Lemma 2.6 in the following.

Lemma 2.6. *Let ρ be as in the proof of Proposition 2.5. Suppose that on $[0,T)$, (2.18) holds. Then*

$$\partial_t\rho+\nabla^*\nabla\rho=-\left|\nabla_{\mathrm{ref}}\psi-\nabla_{\mathrm{ref}}\left(\Pi\psi\right)\right|^2,\qquad\forall\, t\in(0,T).$$

Proof. By standard calculations, we know that

$$\partial_t\rho+\nabla^*\nabla\rho=-\left|\nabla_{\mathrm{ref}}\psi-\nabla_{\mathrm{ref}}\left(\Pi\psi\right)\right|^2+ \tag{2.19}$$

$$+(\psi-\Pi\psi)\cdot\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}\left(\sqrt{g}\,A^j_{\mathrm{ref}}\Pi\psi\right)\right)-(\psi-\Pi\psi)\cdot\nabla^*\nabla\psi+$$

$$+(\psi-\Pi\psi)\cdot\left(A^k_{\mathrm{ref}}\,\partial_k\Pi\psi\right)+(\psi-\Pi\psi)\cdot\left(A^k_{\mathrm{ref}}\,A_{\mathrm{ref},k}\,\Pi\psi\right)+(\psi-\Pi\psi)\cdot\left(\partial_t\psi+\nabla^*_{\mathrm{ref}}\nabla_{\mathrm{ref}}\psi\right).$$

We label from (I) to (VI) the six terms on the right-hand side of (2.19). Now we expand the right-hand side of (2.18) as follows so that we can plug it into the (VI)-th term in (2.19).

$$\partial_t\psi+\nabla^*_{\mathrm{ref}}\nabla_{\mathrm{ref}}\psi=\left(\nu_i,\nabla^*\nabla\Pi\psi\right)\nu_i-\left(\nu_i,A^k_{\mathrm{ref}}\,\partial_k\Pi\psi\right)\nu_i-\left(\nu_i,a^k\,\partial_k\Pi\psi\right)\nu_i- \tag{2.20}$$

$$-\left(\nu_i,A^k_{\mathrm{ref}}\,A_{\mathrm{ref},k}\Pi\psi\right)\nu_i-\left(\nu_i,\frac{1}{\sqrt{g}}\partial_j\left(\sqrt{g}\,A^j_{\mathrm{ref}}\,\Pi\psi\right)\right)\nu_i-\left(\nu_i,a^k\,A_{\mathrm{ref},k}\Pi\psi\right)\nu_i-$$

$$-\left(\nu_i,a^j\,\partial_j\Pi\psi\right)\nu_i-\left(\nu_i,A^k_{\mathrm{ref}}\,a_k\Pi\psi\right)\nu_i-\left(\nu_i,a^k\,a_k\Pi\psi\right)\nu_i+2\,a^k\,\nabla_{\mathrm{ref},k}\Pi\psi+a^ka_k\Pi\psi.$$

We label from (1)-(11) the terms on the right-hand side of (2.20). Notice that

$$(\psi-\Pi\psi)\perp T\mathscr{N}_\psi.$$

Therefore, we can cancel some terms in (2.19) after we plug (2.20) into the (VI)-th term on the right-hand side of (2.19). In fact, (II) and (5), (III) and (1), (IV) and (2), (V) and (4) are the pairs, which can be cancelled out. In (2.20) itself, we see that (3) + (6) + (7) + (8) + (10) give us a tangent vector at ψ. It is orthogonal to $\psi-\Pi\psi$. Obviously, (9) + (11) is also a tangent vector at ψ. Therefore, only the (I)-th term on the right-hand side of (2.19) remains after cancellation. The proof is then finished. □

As a corallary of Proposition 2.5, we can show that the gradient flow (1.3) admits a local regular solution with the initial data (A_0, ϕ_0) given at the beginning of Section II. In fact, set $(a^0, \phi^0) = (A_0 - A_{\rm ref}, \phi_0)$. We can find a smooth solution (a, ψ) of the parabolic gauge equivalent flow by Proposition 2.5. In the rest, one just needs to solve the equation in (2.2) with the initial condition:

$$S(0) = \mathrm{Id}.$$

Obviously,

$$A = S \cdot (A_{\rm ref} + a)\,, \qquad\qquad \phi = S \cdot \psi$$

provides us with a solution of (1.3). Moreover, we have the following energy identity,

Proposition 2.7. *If (A, ϕ) is a regular solution of the gradient flow (1.3) in $[0, T)$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{M}} e(A, \phi)\, \mathrm{d}v_g + \int_{\mathscr{M}} |\partial_t A|^2 + |\partial_t \phi|^2\, \mathrm{d}v_g = 0, \qquad \forall\, t \in (0, T).$$

The proof for this proposition is simple. We inner product $\partial_t A$ and $\partial_t \phi$ on both sides of the first and second equations in (1.3), respectively. Here one may use the fact that $\partial_t \phi$ is orthogonal to the normal vectors $\nu_i(\phi)$. Then integrating by parts, the proof can be achieved.

III. Energy Inequalities and Criterion for First Singular Time

The main purpose of this section is to study a criterion for the first singular time of the gradient flow (1.3). Before that, we consider local energy inequalities, Bochner-type inequality and ϵ-regularity in Section III.1-3, respectively. The criterion will be given in Section III.4. In this section, all balls $B_R(x_0)$ are geodesic balls with $R < i(\mathscr{M})$, where $i(\mathscr{M})$ is the infimum of the injectivity radius of each point $x \in \mathscr{M}$.

III.1. Local Energy Inequalities

In this section, we prove the following local energy inequalities for a solution of the gradient flow (1.3).

Proposition 3.1. *Suppose that (A, ϕ) is a smooth solution of (1.3) in $\mathscr{M} \times [0, T_0)$. Then for all $x_0 \in \mathscr{M}$, $0 < R < i(\mathscr{M})$ and $0 \le S < T < T_0$, we have*

$$\int_{B_{R/2}(x_0)} e(A, \phi)\, \mathrm{d}v_g \bigg|_T \le \int_{B_R(x_0)} e(A, \phi)\, \mathrm{d}v_g \bigg|_S + C\, E(S)\, R^{-2}\, (T - S) \tag{3.1}$$

and

$$\int_{B_{R/2}(x_0)} e(A, \phi)\, \mathrm{d}v_g \bigg|_S \le \int_{B_R(x_0)} e(A, \phi)\, \mathrm{d}v_g \bigg|_T + C\, E(S)\, R^{-2}\, (T - S) + C \int_S^T \int_{\mathscr{M}} |\, \partial_t A\,|^2 + |\, \partial_t \phi\,|^2, \tag{3.2}$$

where $E(S)$ is the total energy of (A, ϕ) at time S and C is independent of x_0, (A, ϕ), R, S and T.

Proof. Choose x_0 and R as in the assumption of Proposition 3.1. Define f a cut-off function such that $f \equiv 1$ in $B_{R/2}(x_0)$, $f \equiv 0$ outside $B_R(x_0)$ and $|f| \leq 1$ on $\mathscr{M}$. Moreover, we assume that $|\nabla f| \leq C/R$, where $C > 0$ is an universal constant.

Inner product $f^2\, \partial_t \phi$ on both sides of the second equation in (1.3) and integrate by parts. We imply that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}} f^2\, |D_A\phi|^2\, \mathrm{d}v_g + \int_{\mathscr{M}} f^2\, |\partial_t \phi|^2\, \mathrm{d}v_g + 2\int_{\mathscr{M}} f\, (D_A\phi, \partial_t\phi\ \mathrm{d}f)\ \mathrm{d}v_g = \int_{\mathscr{M}} \left(D_A\phi, f^2\, \partial_t A\cdot\phi\right)\, \mathrm{d}v_g.$$

Inner product $f^2\, \partial_t A$ on both sides of the first equation in (1.3) and integrate by parts. One has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}} f^2\, |F_A|^2\, \mathrm{d}v_g + \int_{\mathscr{M}} f^2\, |\partial_t A|^2\, \mathrm{d}v_g + 2\int_{\mathscr{M}} f\, \langle F_A, \mathrm{d}f\wedge\partial_t A\rangle\ \mathrm{d}v_g = -\int_{\mathscr{M}} \left(D_A\phi, f^2\, \partial_t A\cdot\phi\right)\, \mathrm{d}v_g.$$

Sum the above two equalities. One can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}} f^2\, e(A,\phi)\, \mathrm{d}v_g + \int_{\mathscr{M}} f^2\left(|\partial_t A|^2 + |\partial_t\phi|^2\right)\mathrm{d}v_g = -2\int_{\mathscr{M}} f\, \langle F_A, \mathrm{d}f\wedge\partial_t A\rangle + f\,(D_A\phi, \partial_t\phi\ \mathrm{d}f)\ \mathrm{d}v_g. \tag{3.3}$$

In one way, by Young's inequality, one knows from (3.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}} f^2\, e(A,\phi)\, \mathrm{d}v_g \leq C\, R^{-2}\int_{\mathscr{M}} e(A,\phi)\, \mathrm{d}v_g.$$

Integrate the above inequality from S to T and apply Proposition 2.7. One has

$$\int_{\mathscr{M}} f^2\, e(A,\phi)\, \mathrm{d}v_g\bigg|_T \leq \int_{\mathscr{M}} f^2\, e(A,\phi)\, \mathrm{d}v_g\bigg|_S + C\, E(S)\, R^{-2}\, (T-S).$$

Notice the choice of the cut-off function f. We know that (3.1) holds. In another way, still by (3.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}} f^2\, e(A,\phi)\, \mathrm{d}v_g + C\int_{\mathscr{M}} f^2\left(|\partial_t A|^2 + |\partial_t\phi|^2\right)\ \mathrm{d}v_g \geq -C\, R^{-2}\int_{\mathscr{M}} e(A,\phi)\, \mathrm{d}v_g.$$

Same as the derivation of (3.1), we can integrate the above inequality from S to T. Then (3.2) holds. □

III.2. Bochner-type Inequality

Proposition 3.2. *Suppose that (A,ϕ) is a regular solution of the gradient flow (1.3). Then*

$$(\partial_t - \Delta_{\mathscr{M}})\, e(A,\phi) + |\nabla_A F_A|^2 + \left|\nabla_A^2\phi\right|^2 \leq C\ \left(|R_{\mathscr{M}}| + |F_A|\right) e(A,\phi) + \left(D_A\nu_i(\phi), D_A\phi\right)^2.$$

Here $\Delta_{\mathscr{M}}$ is the Laplace-Beltrami operator of the manifold $\mathscr{M}$. $C > 0$ is an universal constant depending only on the geometry of $\mathscr{M}$. $R_{\mathscr{M}}$ is the Riemannian curvature of $\mathscr{M}$.

Proof. In one way, it can be shown that (see [5])

$$-\Delta_{\mathscr{M}}\left(\frac{|D_A\phi|^2}{2}\right)=\left(\nabla_A^*\nabla_A D_A\phi, D_A\phi\right)-\left|\nabla_A\left(D_A\phi\right)\right|^2.$$

In another way, by making time derivative once and applying the equation (1.3), we have

$$\partial_t\left(\frac{|D_A\phi|^2}{2}\right)+|(g_l\phi, D_A\phi)|^2=-\left(D_A\left(D_A^*D_A\phi\right), D_A\phi\right)-\left(\left(D_A^*F_A\right)\phi, D_A\phi\right)+\left(D_A\nu_i(\phi), D_A\phi\right)^2.$$

Therefore, sum the above two equalities together,

$$(\partial_t-\Delta_{\mathscr{M}})\left(\frac{|D_A\phi|^2}{2}\right)+|(g_l\phi, D_A\phi)|^2+|\nabla_A(D_A\phi)|^2=$$

$$=\left(D_A\nu_i(\phi), D_A\phi\right)^2-\left(D_A\left(D_A^*D_A\phi\right)-\nabla_A^*\nabla_A D_A\phi, D_A\phi\right)-\left(\left(D_A^*F_A\right)\phi, D_A\phi\right).$$

By Weitzenböck formula, one knows that

$$D_A\left(D_A^*D_A\phi\right)-\nabla_A^*\nabla_A D_A\phi=R_{\mathscr{M}}\times D_A\phi+F_A\times D_A\phi-D_A^*(F_A\phi).$$

Moreover, one can also show that

$$\left(D_A^*F_A\right)\phi=D_A^*(F_A\phi)+*(*F_A\wedge D_A\phi).$$

Therefore,

$$(\partial_t-\Delta_{\mathscr{M}})\left(\frac{|D_A\phi|^2}{2}\right)+|(g_l\phi, D_A\phi)|^2+|\nabla_A(D_A\phi)|^2=$$

$$=\left(D_A\nu_i(\phi), D_A\phi\right)^2-\left(R_{\mathscr{M}}\times D_A\phi+F_A\times D_A\phi, D_A\phi\right)-\left(*\left(*F_A\wedge D_A\phi\right), D_A\phi\right).$$

Obviously, we can bound the right-hand side of the above equality and get

$$(\partial_t-\Delta_{\mathscr{M}})\left(\frac{|D_A\phi|^2}{2}\right)+\left|\nabla_A^2\phi\right|^2\leq\left(D_A\nu_i(\phi), D_A\phi\right)^2+C\left(|R_M|+|F_A|\right)|D_A\phi|^2, \tag{3.4}$$

where $C>0$ depends only on the geometry of $\mathscr{M}$.

As for $|F_A|^2$, we know that

$$\Delta_{\mathscr{M}}\left(\frac{|F_A|^2}{2}\right)=-\left\langle\nabla_A^*\nabla_A F_A, F_A\right\rangle+|\nabla_A F_A|^2.$$

Moreover, by the equation (1.3),

$$\partial_t\left(\frac{|F_A|^2}{2}\right)=-\left\langle D_A\left(D_A^*F_A\right), F_A\right\rangle-\left\langle D_A\left(g_l\phi, D_A\phi\right)g_l, F_A\right\rangle.$$

Therefore,

$$(\partial_t-\Delta_{\mathscr{M}})\left(\frac{|F_A|^2}{2}\right)+|\nabla_A F_A|^2=-\left\langle D_A D_A^*F_A-\nabla_A^*\nabla_A F_A, F_A\right\rangle-\left\langle D_A\left(g_l\phi, D_A\phi\right)g_l, F_A\right\rangle.$$

Apply Bianchi's identity, we have

$$D_A D_A^*F_A-\nabla_A^*\nabla_A F_A=R_{\mathscr{M}}\times F_A+F_A\times F_A.$$

Hence, for suitably large constant C,

$$(\partial_t - \Delta_{\mathscr{M}}) \left(\frac{|F_A|^2}{2} \right) + |\nabla_A F_A|^2 \leq C \left(|R_{\mathscr{M}}| + |F_A| \right) |F_A|^2 - \langle D_A \left(g_l \phi, D_A \phi \right) g_l, F_A \rangle .$$

Note that

$$\langle F_A, D_A \left(g_l \phi, D_A \phi \right) g_l \rangle = 2 \langle F_A, \left(g_l D_A \phi, D_A \phi \right) g_l \rangle + |F_A \phi|^2 .$$

Therefore, one may imply that

$$(\partial_t - \Delta_{\mathscr{M}}) \left(\frac{|F_A|^2}{2} \right) + |\nabla_A F_A|^2 \leq C \left(|R_{\mathscr{M}}| + |F_A| \right) e(A, \phi). \tag{3.5}$$

The proof is then completed by summing (3.4) with (3.5). □

III.3. ϵ-Regularity

We study an ϵ-regularity in this section. In the following, for a given $r > 0$ and $z_0 = (x_0, t_0) \in \mathscr{M} \times \mathbb{R}$, $P_r(z_0)$ denotes the cylinder

$$P_r(z_0) = \left\{ (x, t) \in \mathscr{M} \times \mathbb{R} : x \in B_r(x_0),\ t_0 - r^2 \leq t < t_0 \right\} .$$

If $z_0 = 0$, we simply denote $P_r(0)$ by P_r. Now we state our ϵ-regularity as follows.

Proposition 3.3 (ϵ-regularity)**.** *There exist two positive constants $\delta_0 = \delta_0(m, \mathscr{M})$ and $\epsilon_0 = \epsilon_0(m, \mathscr{M}, \mathscr{N})$ such that if for some*

$$R_0 \in \left(0, \min \left\{ i(\mathscr{M}), T_0^{1/2} \right\} \right) ,$$

we have

$$\sup_{T_0 - R_0^2 \leq t < T_0} \int_{B_{R_0}(x_0)} e(A(t), \phi(t)) \, \mathrm{d}v_g < \epsilon_0, \tag{3.6}$$

then

$$\sup_{P_{R_0/3}(x_0, T_0)} e(A, \phi) \leq 36 \, \delta_0 \, R_0^{-2}.$$

Proof. Our proof follows [5] and [18] with some modifications. For convenience, we divide our arguments into four steps shown below.

Step 1. Choose $t_n \uparrow T_0$ and denote the point (x_0, t_n) by z_n. Obviously, we have $P_{R_0/2}(z_n) \subset P_{R_0}(z_0)$ when n is suitably large. Here $z_0 = (x_0, T_0)$. Let $r_n \in [R_0/4, R_0/2]$ such that

$$(R_0/2 - r_n)^2 \sup_{P_{r_n}(z_n)} e(A, \phi) = \max_{R_0/4 \leq r \leq R_0/2} \left((R_0/2 - r)^2 \sup_{P_r(z_n)} e(A, \phi) \right) .$$

Choose $z_n^* \in \overline{P}_{r_n}(z_n)$ such that

$$e_n := e(A, \phi)(z_n^*) = \sup_{P_{r_n}(z_n)} e(A, \phi).$$

If for some $\delta_0 > 0$, we have

$$e_n \leq \delta_0 \, (R_0/2 - r_n)^{-2} \, , \tag{3.7}$$

then

$$(R_0/2 - R_0/3)^2 \sup_{P_{R_0/3}(z_n)} e(A, \phi) \leq (R_0/2 - r_n)^2 e_n \leq \delta_0.$$

Moreover,

$$\sup_{P_{R_0/3}(z_n)} e(A, \phi) \leq 36 \, \delta_0 \, R_0^{-2}. \tag{3.8}$$

If (3.8) holds for any n suitably large, then the proof can be completed by taking $n \to \infty$. In the following, we show that there are $\delta_0 > 0$ and $\epsilon_0 > 0$ such that when (3.6) holds, (3.7) is true for any n suitably large. Furthermore, (3.8) holds for all n suitably large.

Step 2. If on the contrary that (3.7) fails for some n suitably large. Then

$$\gamma_n := \left(\delta_0 \, e_n^{-1}\right)^{\frac{1}{2}} /2 < (R_0/2 - r_n)/2.$$

Clearly, one may imply that

$$P_{\gamma_n}(z_n^*) \subset P_{(r_n + R_0/2)/2}(z_n). \tag{3.9}$$

Rescale (A, ϕ) in $P_{\gamma_n}(z_n^*)$ by

$$A_n = \gamma_n \, A\left(x_n^* + \gamma_n \, y, t_n^* + \gamma_n^2 \, s\right), \qquad \phi_n = \phi\left(x_n^* + \gamma_n \, y, t_n^* + \gamma_n^2 \, s\right), \qquad (y, s) \in P_1,$$

where $z_n^* = (x_n^*, t_n^*)$. The metric in $B_1(0)$ is induced from g in $B_{\gamma_n}(x_n^*)$ by

$$g_{n,ij}(y) = g_{ij}(x_n^* + \gamma_n \, y), \qquad \forall \, y \in B_1(0).$$

On P_1, we define

$$H_n = \gamma_n^{-2} \, |F_{A_n}|^2 + |\, D_{A_n} \phi_n \,|^2.$$

It is known by the above definitions that

$$H_n(0,0) \geq 2 \gamma_n^2 \, (R_0/2 - r_n)^{-2} \left(R_0/2 - \frac{r_n + R_0/2}{2} \right)^2 \sup_{P_{(r_n + R_0/2)/2}(z_n)} e(A, \phi).$$

Notice (3.9), the definition of γ_n and the rescaling (A_n, ϕ_n), one may imply that

$$\sup_{P_1} H_n \leq 2 \, \delta_0. \tag{3.10}$$

Step 3. Fix $s_0 \in [-1, 0]$. By (3.10) and the regularity of the flow, we have

$$\sup_{B_1} |F_{A_n}|^2(\cdot, s_0) \leq 2 \, \delta_0 \, \gamma_n^2 \leq \delta_0 \, i(\mathscr{M})^2. \tag{3.11}$$

Choose a positive constant $\kappa(m)$ according to Theorem 1.3 in [19] and set $\delta_0 \, i(\mathscr{M})^2 = \kappa(m)$. It is clear that when $\kappa(m)$ is suitably small, we can then find a smooth gauge transformation $S(s_0)$ such that $\mathrm{d} + A_n(\cdot, s_0)$ is gauge equivalent to a connection $\mathrm{d} + A_n^{\mathrm{cg}}(\cdot, s_0)$ which satisfies the Coulomb gauge condition and can be estimated for all $p > m$ as follows:

$$\|A_n^{\mathrm{cg}}(\cdot, s_0)\|_{W^{1,p}(B_1)} \leq c(m) \, \left\|F_{A_n(\cdot, s_0)}\right\|_{L^p(B_1)}. \tag{3.12}$$

Let $\mathcal{O}_{s_0}$ be a neighborhood of s_0 in $[-1,0]$. For any $s \in \mathcal{O}_{s_0}$, we act $S(s_0)$ on the connection $\mathrm{d}+A_n(\cdot,s)$. We denote by $\mathrm{d}+A_n^{\mathrm{cg}}(\cdot,s)$, $s\in\mathcal{O}_{s_0}$, the gauge equivalent connection. Note that even though we put "cg" as a superscript in the gauge equivalent connection, but one should notice that usually only when $s=s_0$, the connection is in Coulomb gauge. By the regularity of the original gradient flow (1.3), we can assume that the length of $\mathcal{O}_{s_0}$ is small enough such that

$$\sup_{s\in\mathcal{O}_{s_0}} \|A_n^{\mathrm{cg}}(\cdot,s)\|_{L^\infty(B_1)} \le \|A_n^{\mathrm{cg}}(\cdot,s_0)\|_{L^\infty(B_1)} + 1.$$

Notice (3.11)-(3.12), we then have by Sobolev embedding theorem that

$$\sup_{s\in\mathcal{O}_{s_0}} \|A_n^{\mathrm{cg}}(\cdot,s)\|_{L^\infty(B_1)} \le c(m),$$

where $c(m)>0$ is a suitably large constant depending on m. It is clear that

$$\{\mathcal{O}_{s_0} : s_0 \in [-1,0]\}$$

forms a covering of $[-1,0]$. Therefore, we can find a set of finite neighborhoods $\{\mathcal{O}_{s_i}\}$ to cover $[-1,0]$ and

$$\max_i \sup_{s\in\mathcal{O}_{s_i}} \|A_n^{\mathrm{cg}}(\cdot,s)\|_{L^\infty(B_1)} \le c(m). \tag{3.13}$$

Step 4. By the rescaling in Step 2, we know that in P_1,

$$\partial_s H_n - \Delta_{g_n} H_n = 2\gamma_n^4\,(\partial_t - \Delta_{\mathscr{M}})\,e(A,\phi).$$

Apply the Bochner-typer inequality in Proposition 3.2 and (3.11), we have

$$\partial_s H_n - \Delta_{g_n} H_n \le C_{m,\mathscr{M}} H_n + 2\left(D_{A_n}\nu_i(\phi_n), D_{A_n}\phi_n\right)^2.$$

Fix an $\mathcal{O}_{s_i}$ in Step 3 and notice that the above inequality is gauge invariant. Therefore,

$$\partial_s H_n - \Delta_{g_n} H_n \le C_{m,\mathscr{M}} H_n + 2\left(D_{A_n^{\mathrm{cg}}}\nu_i(\phi_n^{\mathrm{cg}}), D_{A_n^{\mathrm{cg}}}\phi_n^{\mathrm{cg}}\right)^2.$$

Notice (3.13). We know that there is a positive constant $C_{m,\mathscr{M},\mathscr{N}}$ such that

$$\partial_s H_n - \Delta_{g_n} H_n \le C_{m,\mathscr{M},\mathscr{N}} H_n, \qquad \text{in } \mathcal{O}_{s_i}\times B_1,\ \ \forall\, i.$$

Apply parabolic Harnack inequality (see Theorem 6.17 in [8]). We have

$$\delta_0/2 = H_n(0,0) \le C_{m,\mathscr{M},\mathscr{N}} \int_{P_1} H_n \le C_{m,\mathscr{M},\mathscr{N}}\gamma_n^{-2} \int_{P_{\gamma_n}(z_n^*)} e(A,\phi)\,\mathrm{d}v_g\,\mathrm{d}t. \tag{3.14}$$

Since $P_{\gamma_n}(z_n^*) \subset P_{R_0}(z_0)$, one may imply from (3.14) that

$$\delta_0/2 \le C_{m,\mathscr{M},\mathscr{N}} \sup_{T_0-R_0^2\le t<T_0} \int_{B_{R_0}(x_0)} e(A(t),\phi(t))\,\mathrm{d}v_g \le \epsilon_0\, C_{m,\mathscr{M},\mathscr{N}}.$$

Therefore, when we choose ϵ_0 small enough, then (3.14) fails. In other words, (3.7) holds for any n suitably large, where δ_0 is determined in Step 3. The proof is then finished.

□

III.4. Criterion for First Singular Time

Suppose that (A,ϕ) is a regular solution of (1.3) in $\mathscr{M}\times[0,T_0)$. We claim that

Proposition 3.4. *If for any $x_0\in\mathscr{M}$, we have*

$$\lim_{R\to 0}\limsup_{t\uparrow T_0}\int_{B_R(x_0)} e(A(t),\phi(t))\,\mathrm{d}v_g<\epsilon_1=\epsilon_0/2,$$

where ϵ_0 is determined as in Proposition 3.3, then the solution (A,ϕ) can be smoothly extended across T_0.

Remark 3.5.

(1). From Proposition 3.4, we see that if for some $T_0\in(0,\infty)$, $[0,T_0)$ is a maximal time interval for the solution (A,ϕ), then for some $x_0\in\mathscr{M}$, we must have

$$\limsup_{t\uparrow T_0}\int_{B_R(x_0)} e(A(t),\phi(t))\,\mathrm{d}v_g\geq\epsilon_1,\qquad\forall\,R>0.\tag{3.15}$$

The above inequality provides us with a criterion for the first singular time of the gradient flow (1.3). We call x_0 a singular point at T_0 if (3.15) holds;

(2). The gradient flow (1.3) admits only finitely many singularities. The total number of these singularities is bounded by $2E_0/\epsilon_1$, where E_0 is the initial energy of the flow (1.3).

Proof of (2) in Remark 3.5.

Suppose that $\{x_1,...,x_N\}$ is a set of singular points in $\mathscr{M}$ at T_0. Then for each $i\in\{1,...,N\}$, we can find $t_n^i\uparrow T_0$ (increasing with respect to n) such that

$$\lim_{n\to\infty}\int_{B_\delta(x_i)\times\{t_n^i\}} e(A,\phi)\,\mathrm{d}v_g\geq\epsilon_1/2,\tag{3.16}$$

where

$$\delta=\frac{1}{4}\min\left\{i(\mathscr{M}),\ |x_i-x_j|:i\neq j,\ i,j=1,...,N\right\}.$$

Without loss of generality, we can assume that

$$t_n^i<t_n^{i+1},\qquad\forall\,n\in\mathbb{N}\text{ and }\forall\,i\in\{1,...,N-1\}.$$

Apply the local energy inequality in Proposition 3.1, we know that for any i and n,

$$\int_{B_\delta(x_i)\times\{t_n^i\}} e(A,\phi)\,\mathrm{d}v_g\leq\int_{B_{2\delta}(x_i)\times\{t_n^1\}} e(A,\phi)\,\mathrm{d}v_g+C\,E_0\,(t_n^i-t_n^1)\,\delta^{-2}.$$

Sum the above inequality from $i=1$ to N. We have

$$\sum_{i=1}^N\int_{B_\delta(x_i)\times\{t_n^i\}} e(A,\phi)\,\mathrm{d}v_g\leq\int_{\cup_{i=1}^N B_{2\delta}(x_i)\times\{t_n^1\}} e(A,\phi)\,\mathrm{d}v_g+C\,E_0\,\delta^{-2}\sum_{i=1}^N(t_n^i-t_n^1),\quad\forall\,n\in\mathbb{N}.$$

By Proposition 2.7, the total energy of the gradient flow (1.3) is non-increasing. Therefore,

$$\sum_{i=1}^N\int_{B_\delta(x_i)\times\{t_n^i\}} e(A,\phi)\,\mathrm{d}v_g\leq\int_{\mathscr{M}} e(A_0,\phi_0)\,\mathrm{d}v_g+C\,E_0\,\delta^{-2}\sum_{i=1}^N(t_n^i-t_n^1),\quad\forall\,n\in\mathbb{N}.$$

Notice (3.16). We then send $n \to \infty$ and imply that

$$N\epsilon_1/2 \le \int_{\mathscr{M}} e(A_0, \phi_0)\, \mathrm{d}v_g.$$

Therefore, we know that the total number of the singularities is bounded by $2E_0/\epsilon_1$. □

In the rest of this section, we prove Proposition 3.4. For convenience, we set

$$e_2 = e_2(A, \phi) = |\nabla_A F_A|^2 + |\nabla_A^2 \phi|^2 + 1,$$

where ∇_A is the induced covariant derivative.

Proof of Proposition 3.4 .

Step 1. Uniform boundedness of $e(A, \phi)$ and fine covering of $\mathscr{M}$.

By the assumptions in Proposition 3.4, for any $x_0 \in \mathscr{M}$, we can find a r_0 such that

$$\limsup_{t \uparrow T_0} \int_{B_{r_0}(x_0)} e(A(t), \phi(t))\, \mathrm{d}v_g < \epsilon_1.$$

Furthermore, there is a $T_{x_0} < T_0$ such that

$$\sup_{T_{x_0} \le t < T_0} \int_{B_{r_0}(x_0)} e(A(t), \phi(t))\, \mathrm{d}v_g \le \epsilon_1 < \epsilon_0.$$

Apply Proposition 3.3, we conclude that $e(A, \phi)$ is uniformly bounded in a cylinder $P_{r_1}(x_0, T_0)$ with r_1 sufficiently small. Since $x_0 \in \mathscr{M}$ is arbitrary and $\mathscr{M}$ is compact, then there is a $T_1 < T_0$ sufficiently close to T_0 such that $e(A, \phi)$ is uniformly bounded on

$$P_{T_1, T_0} := \mathscr{M} \times [T_1, T_0).$$

Let $z_0 = (x_0, t_0)$ be an arbitrary point in P_{T_1,T_0}. By Hölder's inequality,

$$\left\| F_{A(t_0)} \right\|_{L^1(B_{R_*}(x_0))} \le C_{\mathscr{M}} R_* \left\| F_{A(t_0)} \right\|_{L^2(B_{R_*}(x_0))}, \qquad \forall\, R_* < i(\mathscr{M}).$$

By Proposition 2.7, the energy $E(A, \phi)$ is non-increasing. Therefore,

$$\left\| F_{A(t_0)} \right\|_{L^1(B_{R_*}(x_0))} \le C_{\mathscr{M}}\, E_0^{1/2}\, R_*, \qquad \forall\, R_* < i(\mathscr{M}).$$

Hence, we can take R_* suitably small which depends on the geometry of $\mathscr{M}$ and E_0 such that by Theorem 1.3 in [19], $A(t_0)$ is gauge equivalent to a connection $A^{\mathrm{cg}}(t_0)$ on $B_{R_*}(x_0)$. $A^{\mathrm{cg}}(t_0)$ satisfies the Coulomb gauge condition and can be estimated as follows:

$$\|A^{\mathrm{cg}}(t_0)\|_{W^{1,p}(B_{R_*}(x_0))} \le C_{\mathscr{M}} \|F_{A(t_0)}\|_{L^p(B_{R_*}(x_0))}, \qquad \forall\, p > 2.$$

Let $p \to \infty$. We know that on $B_{R_*}(x_0)$, the $W^{1,\infty}$-norm of $A^{\mathrm{cg}}(t_0)$ is uniformly bounded. Motivated by the above discussions, we fix a finite covering of $\mathscr{M}$, denoted by $\Sigma' = \{B_{R_*}(y_i)\}$. The total number of geodesic balls in Σ' can be bounded by a constant depending on the geometry of $\mathscr{M}$ and E_0. We refer Σ' as a fine covering of $\mathscr{M}$.

Step 2. Follow the similar arguments as in Section III.2, one can show that in P_{T_1,T_0},

$$(\partial_t - \Delta_{\mathscr{M}})\, e_2 + 2\, e_3 \le C\, e_2 + \mathrm{Rem}, \tag{3.17}$$

where $C > 0$ is a constant independent of $(x,t) \in P_{T_1,T_0}$,

$$e_3 = e_3(A,\phi) := |\nabla_A^2 F_A|^2 + |\nabla_A^3 \phi|^2,$$

and "Rem" is the sum of the following six terms.

$$I = \left| \left(\nabla_A^2 N_\phi, \nabla_A^2 \phi\right) \right|, \qquad \text{where } N_\phi = (D_A\phi, D_A\nu_i(\phi))\, \nu_i(\phi);$$

$$II = |\nabla_A H\phi|^2, \quad III = |\nabla_A D_A H|^2, \qquad \text{where } H = (g_l\phi, D_A\phi)\, g_l \text{ and } H\phi \in \Omega^0\left(T^*\mathscr{M} \otimes \phi^* T\mathscr{N}\right);$$

$$IV = |\nabla_A (R_{\mathscr{M}} \times D_A\phi + F_A \times D_A\phi)|^2, \; V = |\nabla_A (*(*F_A \wedge D_A\phi))|^2, \; VI = |\nabla_A (R_{\mathscr{M}} \times F_A + F_A \times F_A)|^2.$$

Note that in the above six terms, we omit some useless positive coefficients. Take $p > 2$ and multiply e_2^{p-1} on both sides of (3.17). Integrate over $\mathscr{M}$. We have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}} e_2^p \,\mathrm{d}v_g + 2\int_{\mathscr{M}} e_2^{p-1} e_3 \,\mathrm{d}v_g \le C\int_{\mathscr{M}} e_2^p \,\mathrm{d}v_g + \int_{\mathscr{M}} e_2^{p-1}\cdot \mathrm{Rem}\,\mathrm{d}v_g. \tag{3.18}$$

In the last integral of (3.18), there are six terms by the definition of Rem. One can check that the integral of e_2^{p-1} multiplied by II to VI can be absorbed by the first integral on the right-hand side of (3.18). We only need to study the integral of e_2^{p-1} multiplied by I. Recall the fine covering Σ' of $\mathscr{M}$ in Step 1. We have

$$\int_{\mathscr{M}} e_2^{p-1}\cdot I \,\mathrm{d}v_g \le \sum_i \int_{B_{R_*}(y_i)} e_2^{p-1}\cdot I\,\mathrm{d}v_g. \tag{3.19}$$

Step 3. Fix an arbitrary $t \in [T_1, T_0)$ and focus our study on one geodesic ball $B_i := B_{R_*}(y_i) \in \Sigma'$. Notice the fact that I and e_2 both are gauge invariant. Therefore, we can assume, by the discussions in Step 1, that at time t and in ball B_i, (A,ϕ) is in good gauge with $A(t)$ uniformly bounded in $W^{1,\infty}(B_i)$. Notice that $e(A,\phi)$ is uniformly bounded on P_{T_1,T_0}. Hence, in this good gauge, $\phi(t)$ is also uniformly bounded in $W^{1,\infty}(B_i)$. Meanwhile, one may imply that the L^∞-norm of $D_A\nu_i(\phi)$ is uniformly bounded on B_i. As for the higher order covariant derivatives of $\nu_i(\phi)$ at time t, we have

Lemma 3.6. *Fix an arbitrary $t \in [T_1, T_0)$ and suppose that on B_i, $(A(t),\phi(t))$ is in the good gauge. Then at time t, we can find bounded quantities K_i $(i = 1, ..., 4)$ such that the following decompositions hold:*

$$\nabla_A^2 \nu_i(\phi) = L_{\phi,i}^{\mathscr{N}}(\phi)\,\nabla_A^2\phi + K_1, \qquad \nabla_A^3 \nu_i(\phi) = L_{\phi,i}^{\mathscr{N}}(\phi)\,\nabla_A^3\phi + K_2 \times \nabla_A^2\phi + K_3 \times \nabla^2 A + K_4.$$

Here

$$L_{\phi,i}^{\mathscr{N}} = \nabla\left(\nu_i \circ \Pi\right)(\phi)$$

is a matrix with Π the projection from $\mathscr{N}_\delta$ onto $\mathscr{N}$. K_i $(i = 1, ..., 4)$ depend on the following uniformly bounded quantities: ϕ, $\nu_i(\phi)$, the curvature F_A, the covariant derivatives $D_A\phi$ and $D_A\nu_i(\phi)$, the connection A and the first derivatives of A.

The proof of Lemma 3.6 can be carried out by straightforward calculations. We omit it here.

Step 4. Now, we estimate $\int_{B_i} e_2^{p-1}\cdot I\,\mathrm{d}v_g$. Note that (A,ϕ) is assumed to be in the good gauge discussed as above. By calculations, one can imply that

$$\nabla_A^2 N_\phi = \nabla^2\left(D_A\phi, D_A\nu_i(\phi)\right)\nu_i(\phi) +$$

$$+ d\left(D_A\phi, D_A\nu_i(\phi)\right) \otimes \nabla_A\phi + \nabla_A\phi \otimes d\left(D_A\phi, D_A\nu_i(\phi)\right) + \left(D_A\phi, D_A\nu_i(\phi)\right)\nabla_A^2\nu_i(\phi).$$

Label from N_1 to N_4 the four terms on the right-hand side above. Therefore, we have

1. N_4. By the uniform boundedness of $D_A\phi$ and $D_A\nu_i(\phi)$, one can imply from Lemma 3.6 that

$$\left|\left(N_4, \nabla_A^2\phi\right)\right| \le C\left|\nabla_A^2\phi\right|\left|\nabla_A^2\nu_i(\phi)\right| \le C\,e_2; \tag{3.20}$$

2. N_2 and N_3. Since A is metric,

$$d\left(D_A\phi, D_A\nu_i(\phi)\right) = \left(\nabla_A^2\phi, D_A\nu_i(\phi)\right) + \left(D_A\phi, \nabla_A^2\nu_i(\phi)\right).$$

Similarly as in the case of N_4, one has

$$\left|\left(N_2, \nabla_A^2\phi\right)\right| + \left|\left(N_3, \nabla_A^2\phi\right)\right| \le C\left|\nabla_A^2\phi\right|^2 + C\left|\nabla_A^2\phi\right|\left|\nabla_A^2\nu_i(\phi)\right| \le C\,e_2; \tag{3.21}$$

3. N_1. Note that $\nu_i(\phi)$ is orthogonal to $\nabla_{A,j}\phi$, where $\nabla_{A,j}\phi = \partial_j\phi + A_j\phi$. Therefore,

$$(\nu_i(\phi), \nabla_A^2\phi) = -(\nabla_{A,i}\phi, \nabla_{A,k}\nu_i(\phi))\,\mathrm{d}x^i \otimes \mathrm{d}x^k.$$

Apply the uniform boundedness of $D_A\phi$ and $D_A\nu_i(\phi)$. We know that

$$|(N_1, \nabla_A^2\phi)| \le C\,|\nabla^2(D_A\phi, D_A\nu_i(\phi))| \le C\,(|\nabla_A^3\phi| + |\nabla_A^3\nu_i(\phi)|) + C\,e_2.$$

In light of the decomposition for $\nabla_A^3\nu_i(\phi)$ in Lemma 3.6, one can show that

$$|\nabla_A^3\nu_i(\phi)| \le C(|\nabla_A^3\phi| + |\nabla_A^2\phi| + |\nabla^2 A| + 1).$$

Moreover,

$$|(N_1, \nabla_A^2\phi)| \le C(|\nabla_A^3\phi| + |\nabla^2 A|) + C\,e_2. \tag{3.22}$$

Notice (3.20)-(3.22). On B_i, I is bounded by the right-hand side of (3.22) with suitably large constant C.

Now we apply all the arguments above to (3.18). Hence, by (3.19), we know that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}} e_2^p\,\mathrm{d}v_g + 2\int_{\mathscr{M}} e_2^{p-1}\,e_3\,\mathrm{d}v_g \le C\int_{\mathscr{M}} e_2^p\,\mathrm{d}v_g + C\sum_i\int_{B_i} e_2^{p-1}\left(|\nabla_A^3\phi| + |\nabla^2 A|\right)\mathrm{d}v_g. \tag{3.23}$$

By Young's inequality,

$$\int_{B_i} e_2^{p-1}|\nabla_A^3\phi|\,\mathrm{d}v_g \le \epsilon\int_{B_i} e_2^{p-1}|\nabla_A^3\phi|^2\,\mathrm{d}v_g + C(\epsilon)\int_{B_i} e_2^{p-1}\,\mathrm{d}v_g. \tag{3.24}$$

Notice that the total number of geodesic balls in Σ' is bounded by a constant which depends on the geometry of $\mathscr{M}$ and E_0. Therefore, when ϵ is suitably small, the first term on the right-hand side of (3.24) can be absorbed by the second term on the left-hand side of (3.23). The second term on the right-hand side of (3.24) can be combined into the first term on the right-hand side of (3.23) with suitably large constant C. We are left to study

$$\int_{B_i} e_2^{p-1}\,|\nabla^2 A|\,\mathrm{d}v_g.$$

With minor modifications of Lemma 2.3.11 in [3], one can show that if R_* is suitably small, then in the good gauge discussed above, we have

$$\|A\|_{W^{2,p}(B_i)} \le C_{\mathscr{M}}\left(\|F_A\|_{L^\infty(B_i)} + \|\nabla_A F_A\|_{L^p(B_i)}\right).$$

Therefore, by Hölder's inequality,

$$\int_{B_i} e_2^{p-1} |\nabla^2 A| \,\mathrm{d}v_g \leq \|e_2\|^{p-1}_{L^p(B_i)} \left\|\nabla^2 A\right\|_{L^p(B_i)} \leq C\,\|e_2\|^{p-1}_{L^p(B_i)} \left(1 + \|\nabla_A F_A\|_{L^p(B_i)}\right) \leq C \int_{B_i} e_2^p \,\mathrm{d}v_g + C.$$

Finally, all the above arguments imply that there exists a constant C suitably large, by which

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{M}} e_2^p \,\mathrm{d}v_g \leq C \int_{\mathscr{M}} e_2^p \,\mathrm{d}v_g + C.$$

Solve the above inequality, one knows that,

$$\|e_2(t)\|_{L^p(\mathscr{M})} \leq e^{CT_0}\|e_2(T_1)\|_{L^p(\mathscr{M})} + e^{CT_0}, \qquad \forall\, t \in [T_1, T_0).$$

Let $p \to \infty$. We obtain the uniform boundedness of e_2 on P_{T_1,T_0}.

Step 5. Choose a sequence $t_n \uparrow T_0$. Hence, by the above discussions, $e_2(t_n)$ is uniformly bounded. We can then go through Uhlenbeck's theorem in [19] to find a set of gauge transformations $\{g_n\}$ such that

$$g_n \cdot (A(t_n), \phi(t_n)) \tag{3.25}$$

are uniformly bounded in $W^{2,p}$, for any $p \in (2, \infty)$. Apply the local existence result in Section II. A new solution can be found by solving the gradient flow (1.3) with the initial data (3.25) at time t_n. Moreover, we know that this new solution exists in an interval $[t_n, T)$ with $T > T_0$, provided that t_n is close to T_0. Note that the gradient flow (1.3) is invariant under a time-independent gauge transformation. As a consequence, we can act g_n^{-1} on this new solution and obtain an extension of (A, ϕ) on $[t_n, T)$. The proof is then completed. □

IV. Bubbling Analysis

We consider the bubbling phenomenon associated with (1.3). Throughout this section, $(A(t), \phi(t))$ is a smooth solution of (1.3) on $[0, T_0)$, where $T_0 < \infty$ is its first singular time. For simplicity, we assume that there is only one singular point, denoted by $x_0 \in \mathscr{M}$, at T_0.

IV.1. Convergence of the gradient flow

In this section, as $t \uparrow T_0$, we study the convergence of $(A(t), \phi(t))$ away from the singular point x_0. Suppose that $x_1 \in \mathscr{M} \setminus \{x_0\}$. By the definition of singular point in part (1) of Remark 3.5, one can find a $r_1 > 0$ such that

$$B_{r_1}(x_1) \subset\subset \mathscr{M} \setminus \{x_0\}$$

and moreover,

$$\limsup_{t\uparrow T_0} \int_{B_{r_1}(x_1)} e(A(t), \phi(t)) \,\mathrm{d}v_g < \epsilon_1. \tag{4.1}$$

In light of (4.1), we have

$$\sup_{t\in[T_1,T_0)} \int_{B_{r_1}(x_1)} e(A(t), \phi(t)) \,\mathrm{d}v_g \leq \epsilon_1, \qquad \text{for some } T_1 < T_0.$$

Therefore, by the ϵ-regularity in Proposition 3.3, we imply that

$$\sup_{P_{r_2}(x_1,T_0)} e(A,\phi) \le C\, r_2^{-2}, \qquad \text{for some } r_2 \in (0, r_1) \text{ small enough.} \tag{4.2}$$

Now we consider the uniform boundedness of e_2 which is defined in Section III.4.

Step 1. L^1-integrability for e_2.

By (4.2) and the Bochner-type inequality in Proposition 3.2, we know that

$$(\partial_t - \Delta_{\mathscr{M}})\, e(A,\phi) + e_2 \le C, \qquad \text{on } P_{r_2}(x_1,T_0), \tag{4.3}$$

where $C > 0$ is a constant independent of t. Let η be a non-negative cut-off function such that

$$\eta \equiv 1, \quad \text{on } B_{r_2/2}(x_1); \qquad \eta \equiv 0, \quad \text{outside } B_{r_2}(x_1).$$

If we multiply η on both sides of (4.3) and integrate over $B_{r_2}(x_1)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{r_2}(x_1)} \eta\, e(A,\phi)\,\mathrm{d}v_g - \int_{B_{r_2}(x_1)} \eta\,\Delta_{\mathscr{M}} e(A,\phi)\,\mathrm{d}v_g + \int_{B_{r_2/2}(x_1)} e_2\,\mathrm{d}v_g \le C, \quad \forall\ t \in \left[T_0 - r_2^2, T_0\right).$$

Apply integration by parts twice for the second term on the left-hand side of the above inequality. One has

$$\int_{B_{r_2}(x_1)} \eta\,\Delta_{\mathscr{M}} e(A,\phi)\,\mathrm{d}v_g = \int_{B_{r_2}(x_1)} e(A,\phi)\,\Delta_{\mathscr{M}}\eta\,\mathrm{d}v_g.$$

Therefore, by (4.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{r_2}(x_1)} \eta\, e(A,\phi)\,\mathrm{d}v_g + \int_{B_{r_2/2}(x_1)} e_2\,\mathrm{d}v_g \le C, \qquad \forall\ t \in \left[T_0 - r_2^2, T_0\right).$$

Integrate the above inequality from $T_0 - r_2^2/4$ to T, where $T \in \left[T_0 - r_2^2/4, T_0\right)$. Then by (4.2),

$$\int_{T_0 - r_2^2/4}^{T}\int_{B_{r_2/2}(x_1)} e_2\,\mathrm{d}v_g\,\mathrm{d}t \le C,$$

where C is independent of T. Take $T \uparrow T_0$. We get the desired L^1-integrability for e_2. That is

$$\int_{P_{r_2/2}(x_1,T_0)} e_2\,\mathrm{d}v_g\,\mathrm{d}t \le C. \tag{4.4}$$

Step 2. L^2-integrability for e_2.

In this step, we restrict our study on $P_{r_3}(x_1,T_0)$, where $r_3 = r_2/2$. By Uhlenbeck's theorem, $A(t)$ is gauge equivalent to a Coulomb connection $A^*(t)$ on $B_{r_3}(x_1)$, for any $t \in [T_0 - r_3^2, T_0)$. Notice (4.2) and (4.4). $A^*(t)$ can be estimated as follows:

$$\sup_{t\in[T_0-r_3^2,T_0)} \|A^*(t)\|_{1,\infty;B_{r_3}(x_1)} + \int_{P_{r_3}(x_1,T_0)} \left|\nabla^2 A^*(t)\right|^2\,\mathrm{d}v_g\,\mathrm{d}t \le C. \tag{4.5}$$

By (3.17) and the arguments in Step 2-4 of the proof of Proposition 3.4, one can show that

$$(\partial_t - \Delta_{\mathscr{M}})\, e_2 \le C_2\, e_2 + C_2\,\left|\nabla^2 A^*\right|, \qquad \text{in } P_{r_3}(x_1,T_0).$$

If we define f to be the unique solution for the following initial-boundary-value problem:

$$\begin{cases} (\partial_t - \Delta_{\mathscr{M}})\, f = C_2\, f + C_2\, \left|\nabla^2 A^*\right|, & \text{in } P_{r_3}(x_1, T_0); \\ f = 0, & \text{on } \bar{\partial}\, P_{r_3}(x_1, T_0), \end{cases} \tag{4.6}$$

where $\bar{\partial}\, P_{r_3}(x_1, T_0)$ is the parabolic boundary of $P_{r_3}(x_1, T_0)$. Then obviously,

$$(\partial_t - \Delta_{\mathscr{M}})\, (e_2 - f) \le C_2\, (e_2 - f)\,, \qquad \text{on } P_{r_3}(x_1, T_0).$$

Apply (4.4) and the parabolic Harnack inequality. One can show that

$$\sup_{P_{r_3/2}(x_1,T_0)} (e_2 - f) \le C \int_{P_{r_3}(x_1,T_0)} e_2 + |f|\; \mathrm{d}v_g\, \mathrm{d}t \le C + C \int_{P_{r_3}(x_1,T_0)} |f|\; \mathrm{d}v_g\, \mathrm{d}t.$$

Therefore,

$$e_2 \le |f| + C + C \int_{P_{r_3}(x_1,T_0)} |f|\; \mathrm{d}v_g\, \mathrm{d}t, \qquad \text{on } P_{r_3/2}(x_1, T_0). \tag{4.7}$$

By (4.5)-(4.6), f can be estimated by

$$\sup_{t \in \left[T_0 - r_3^2, T_0\right)} \|f\|_{1,2;B_{r_3}(x_1)} \le C \left\|\nabla^2 A^*\right\|_{2;P_{r_3}(x_1,T_0)} \le C.$$

Apply this estimate in (4.7). One can imply that

$$\sup_{t \in \left[T_0 - r_3^2/4, T_0\right)} \int_{B_{r_3/2}(x_1)} e_2^2\, \mathrm{d}v_g \le C + C \sup_{t \in \left[T_0 - r_3^2, T_0\right)} \int_{B_{r_3}(x_1)} |f|^2\, \mathrm{d}v_g \le C. \tag{4.8}$$

Step 3. L^∞-Boundedness of e_2.

In this step, we restrict our attention on $P_{r_4}(x_1, T_0)$, where $r_4 = r_3/2$. Similarly as in Step 2, $A(t)$ is gauge equivalent to a Coulomb connection $A^{**}(t)$ on $B_{r_4}(x_1)$, for any $t \in [T_0 - r_4^2/2, T_0)$. With (4.8), we have a better estimate for $A^{**}(t)$. That is

$$\sup_{t \in \left[T_0 - r_4^2, T_0\right)} \|A^{**}(t)\|_{1,\infty;B_{r_4}(x_1)} + \sup_{t \in \left[T_0 - r_4^2, T_0\right)} \int_{B_{r_4}(x_1)} \left|\nabla^2 A^{**}(t)\right|^4\, \mathrm{d}v_g \le C. \tag{4.9}$$

Same as in Step 2, we have

$$(\partial_t - \Delta_{\mathscr{M}})\, e_2 \le C_2\, e_2 + C_2\, \left|\nabla^2 A^{**}\right|, \qquad \text{in } P_{r_4}(x_1, T_0).$$

Meanwhile, we define h to be the unique solution for the following initial-boundary-value problem:

$$\begin{cases} (\partial_t - \Delta_{\mathscr{M}})\, h = C_2\, h + C_2\, \left|\nabla^2 A^{**}\right|, & \text{in } P_{r_4}(x_1, T_0); \\ h = 0, & \text{on } \bar{\partial}\, P_{r_4}(x_1, T_0), \end{cases} \tag{4.10}$$

Therefore, by parabolic Harnack inequality and similar arguments as in Step 2, one can imply that

$$e_2 \le |h| + C, \qquad \text{on } P_{r_4/2}(x_1, T_0).$$

Obviously, the L^∞-norm of h is finite on $P_{r_4/2}(x_1, T_0)$ due to (4.9) and Theorem 7.32, Theorem 7.36 in [8]. Therefore, the L^∞-norm of e_2 is finite on $P_{r_4/2}(x_1, T_0)$.

Keep applying similar arguments as above. We conclude that

Proposition 4.1. *For any given $x_1 \in \mathscr{M} \setminus \{x_0\}$, there exists a sequence of decreasing radius $\{s_j\}$ such that*

$$B_{s_j}(x_1) \subset\subset \mathscr{M} \setminus \{x_0\}, \qquad \text{for all } j \in \mathbb{N}$$

and meanwhile, $|\nabla_A^j F_A|^2 + |\nabla_A^{j+1}\phi|^2$ is L^∞-bounded on $P_{s_j}(x_1, T_0)$.

By Proposition 4.1, one can further imply that

Proposition 4.2. *There is (A_*, ϕ_*) smooth away from x_0 so that for any $k \in \mathbb{N}$,*

$$(A(t), \phi(t)) \longrightarrow (A_*, \phi_*), \qquad \text{in } C^k_{\text{loc}}(\mathscr{M} \setminus \{x_0\}), \quad \text{as } t \uparrow T_0.$$

In the following, we define $(A(T_0), \phi(T_0)) = (A_*, \phi_*)$ which is the extension of the gradient flow (1.3) at T_0.

We should understand the convergence in Proposition 4.2 in the following way. For any $x_1 \neq x_0$, one can find a small ball $\mathscr{B}$ so that $x_1 \in \mathscr{B}$ and $x_0 \notin \overline{\mathscr{B}}$. Meanwhile, $\mathscr{B}$ is contained in any $\mathscr{U}_\alpha$ which contains x_1. Here $\{\mathscr{U}_\beta\}$ is the covering that we use to define the principal $\mathscr{G}$-bundle $\mathscr{P}$. Let $(A_\alpha(t), \phi_\alpha(t))$ and $(A_{\alpha,*}, \phi_{\alpha,*})$ be local representations of $(A(t), \phi(t))$ and (A_*, ϕ_*) on $\mathscr{U}_\alpha$, respectively. Hence, Proposition 4.2 implies that for any $k \in \mathbb{N}$,

$$(A_\alpha(t), \phi_\alpha(t)) \longrightarrow (A_{\alpha,*}, \phi_{\alpha,*}), \qquad \text{strongly in } C^k(\overline{\mathscr{B}}), \ \text{ as } t \uparrow T_0.$$

In a word, the convergence in Proposition 4.2 is the convergence for any representation of $(A(t), \phi(t))$. The proof for Proposition 4.2 is trivial. One just needs the equation (1.3). We omit the arguments here.

IV.2. Bubbling Analysis

We start our bubbling analysis. Firstly, we consider an energy identity. Note that for any $\delta > 0$ and $t < T_0$,

$$\int_{B_\delta(x_0)} e(t)\, \mathrm{d}v_g + \int_{\mathscr{M} \setminus B_\delta(x_0)} e(t)\, \mathrm{d}v_g = \int_{\mathscr{M}} e(t)\, \mathrm{d}v_g,$$

where $e(t)$ is a simplified notation for the energy density $e(A(t), \phi(t))$. By Proposition 4.2, when $t \uparrow T_0$,

$$\lim_{t \uparrow T_0} \int_{B_\delta(x_0)} e(t)\, \mathrm{d}v_g + \int_{\mathscr{M} \setminus B_\delta(x_0)} e(T_0)\, \mathrm{d}v_g = \lim_{t \uparrow T_0} \int_{\mathscr{M}} e(t)\, \mathrm{d}v_g.$$

Send $\delta \to 0$. We have the following energy identity:

$$\lim_{t \uparrow T_0} \int_{\mathscr{M}} e(t)\, \mathrm{d}v_g = \int_{\mathscr{M}} e(T_0)\, \mathrm{d}v_g + E_{\text{bubble}}, \tag{4.11}$$

where

$$E_{\text{bubble}} = \lim_{\delta \to 0} \lim_{t \uparrow T_0} \int_{B_\delta(x_0)} e(t)\, \mathrm{d}v_g. \tag{4.12}$$

Notice (3.15). E_{bubble} has a positive lower bound. Therefore, by (4.11), we lose some energy at T_0 due to the existence of the singular point x_0. To recover these lost energy is the main topic of this section. Throughout the following arguments, $B_{r_0}(x_0)$ is a geodesic ball around x_0. r_0 can be adjusted small enough. We choose normal coordinates in $B_{r_0}(x_0)$ so that

$$|\, g_{ij}(x) - \delta_{ij} \,| \leq C\, |x - x_0|^2, \qquad |\, \mathrm{d}\, g_{ij} \,| \leq C\, |x - x_0|, \qquad \forall\, x \in B_{r_0}(x_0). \tag{4.13}$$

1. Bubbling Sequence.

By (4.12), one can find $\delta_n \downarrow 0$ and $t_n \uparrow T_0$ such that

$$E_{\text{bubble}} = \lim_{n\to\infty} \int_{B_{\delta_n}(x_0)} e(t_n)\, \mathrm{d}v_g. \tag{4.14}$$

With $\{\delta_n\}$ and $\{t_n\}$ above, we define two cylinders

$$P_n = B_{r_0}(x_0) \times [\, t_n - 2\delta_n^2, t_n] \qquad \text{and} \qquad P_n^* = B_{r_0\,\delta_n^{-1}} \times [\,-2, 0\,],$$

where $B_{r_0\,\delta_n^{-1}}$ is a ball with center 0 and radius $r_0\,\delta_n^{-1}$. Set

$$A_n = \delta_n A\left(x_0 + \delta_n y, t_n + \delta_n^2 s\right), \qquad \phi_n = \phi\left(x_0 + \delta_n y, t_n + \delta_n^2 s\right), \qquad \forall\ (y,s) \in P_n^*.$$

By the above definitions, one can show that

$$\int_{P_n^*} |\partial_s \phi_n|^2 + \delta_n^{-2} |\partial_s A_n|^2\, \mathrm{d}v_{g_n}\, \mathrm{d}s = \int_{P_n} |\partial_t \phi|^2 + |\partial_t A|^2\, \mathrm{d}v_g\, \mathrm{d}t \longrightarrow 0, \quad \text{as } n \to \infty,$$

where g_n is the rescaled metric defined by

$$g_n(\cdot) = g\left(x_0 + \delta_n \cdot\right), \qquad \text{on } B_{r_0\,\delta_n^{-1}}.$$

Therefore, we can find $s_0 \in [-1, -1/2]$ such that the rescaled kinetic energy satisfies

$$\int_{B_{r_0\,\delta_n^{-1}}} |\partial_s \phi_n|^2(\cdot, s_0) + \delta_n^{-2} |\partial_s A_n|^2(\cdot, s_0)\, \mathrm{d}v_{g_n} \longrightarrow 0, \quad \text{as } n \to \infty. \tag{4.15}$$

For convenience, we define

$$\tau_n = t_n + \delta_n^2\, s_0$$

and set

$$A_{n,s} = A_n(\cdot, s), \qquad \phi_{n,s} = \phi_n(\cdot, s), \qquad \forall\ s \in [s_0, s_n),$$

where $s_n := (T_0 - t_n)\,\delta_n^{-2}$. Particularly, when $s = s_0$, we call $(A_{n,s_0}, \phi_{n,s_0})$ a bubbling sequence.

Choose a sequence $R_k \uparrow \infty$ and denote by B_k the ball $B_{R_k}(0)$. Fix k. When δ_n is small enough, we have

$$\int_{\mathscr{M}} e(t)\, \mathrm{d}v_g \bigg|_{t=\tau_n} \geq \int_{\mathscr{M}\setminus B_{r_0}(x_0)} e(t)\, \mathrm{d}v_g \bigg|_{t=\tau_n} + \int_{B_{\delta_n R_k}(x_0)} e(t)\, \mathrm{d}v_g \bigg|_{t=\tau_n}.$$

Apply the local energy inequality (3.1) in Proposition 3.1. One can show that

$$\int_{B_{\delta_n R_k}(x_0)} e(t)\, \mathrm{d}v_g \bigg|_{t=\tau_n} \geq \int_{B_{\delta_n\, R_k/2}(x_0)} e\left(t_n\right)\, \mathrm{d}v_g + C\, E_0\, s_0\, R_k^{-2}.$$

Therefore, for R_k suitably large, the above two inequalities imply that

$$\int_{\mathscr{M}} e(t)\, \mathrm{d}v_g \bigg|_{t=\tau_n} \geq \int_{\mathscr{M}\setminus B_{r_0}(x_0)} e(t)\, \mathrm{d}v_g \bigg|_{t=\tau_n} + \int_{B_{\delta_n R_k}(x_0)} e(t)\, \mathrm{d}v_g \bigg|_{t=\tau_n} \geq \tag{4.16}$$

$$\geq \int_{\mathscr{M}\setminus B_{r_0}(x_0)} e(t)\,\mathrm{d}v_g \Big|_{t=\tau_n} + \int_{B_{\delta_n}(x_0)} e\left(t_n\right)\,\mathrm{d}v_g + C\,E_0\,s_0\,R_k^{-2}. \tag{4.17}$$

By (4.11),

$$\lim_{n\to\infty} \int_{\mathscr{M}} e(t)\mathrm{d}v_g \Big|_{t=\tau_n} = \int_{\mathscr{M}} e\left(T_0\right)\,\mathrm{d}v_g + E_{\mathrm{bubble}}. \tag{4.18}$$

By (4.14) and Proposition 4.2,

$$\lim_{r_0\to 0}\lim_{k\to\infty}\lim_{n\to\infty} (4.17) = \int_{\mathscr{M}} e\left(T_0\right)\,\mathrm{d}v_g + E_{\mathrm{bubble}}. \tag{4.19}$$

Note that

$$2\int_{B_{\delta_n R_k}(x_0)} e(t)\,\mathrm{d}v_g \Big|_{t=\tau_n} = \int_{B_k} \delta_n^{-2}\left|F_{A_{n,s_0}}\right|^2\,\mathrm{d}v_{g_n} + \int_{B_k}\left|D_{A_{n,s_0}}\phi_{n,s_0}\right|^2\mathrm{d}v_{g_n}. \tag{4.20}$$

Hence from (4.16)-(4.20), to recover the lost energy E_{bubble} relies on the study of the convergence of

$$E_{n,k} := \int_{B_k} \delta_n^{-2}\left|F_{A_{n,s_0}}\right|^2\,\mathrm{d}v_{g_n} + \int_{B_k}\left|D_{A_{n,s_0}}\phi_{n,s_0}\right|^2\mathrm{d}v_{g_n},$$

which is the rescaled energy for the bubbling sequence. In the following, we study the convergence of the rescaled energy for gauge fields. That is the first term in $E_{n,k}$.

2. Gauge Adjustment.

Fix an arbitrary $R>0$. Choose δ_n small enough so that $R\delta_n < r_0$. One then can show that

$$\int_{B_R}\left|F_{A_{n,s_0}}\right|^2\,\mathrm{d}v_{g_n} = \delta_n^2\int_{B_{R\,\delta_n}(x_0)}\left|F_A\right|^2\mathrm{d}v_g\Big|_{\tau_n} \leq C\,E_0\,\delta_n^2. \tag{4.21}$$

Apply Uhlenbeck's theorem. We can find a gauge transformation σ_n such that on B_R,

$$A^*_{n,s_0} := \sigma_n\cdot A_{n,s_0}$$

satisfies the Coulomb gauge condition and moreover,

$$\left\|A^*_{n,s_0}\right\|^2_{2;R} + R^2\,\left\|\nabla A^*_{n,s_0}\right\|^2_{2;R} \leq C\,R^2\,\left\|F_{A_{n,s_0}}\right\|^2_{2;R}, \tag{4.22}$$

where $\|\cdot\|_{2;R}$ stands for the usual $L^2\left(B_R\right)$-norm. We can also define on B_R,

$$\left(A^*_n,\phi^*_n\right)(\cdot,s) = \left(A^*_{n,s},\phi^*_{n,s}\right) = \sigma_n\cdot\left(A_{n,s},\phi_{n,s}\right), \qquad \forall\ s\in[s_0,s_n).$$

Since (1.3) is invariant under time-independent gauge transformation, (A^*_n,ϕ^*_n) must satisfy

$$\partial_s A^*_n = -D^*_{A^*_n}F_{A^*_n} - \delta_n^2\,\left(g_l\,\phi^*_n, D_{A^*_n}\phi^*_n\right)\,g_l, \qquad \text{on } B_R\times[s_0,s_n), \tag{4.23}$$

where in (4.23), we are using the rescaled metric g_n. Particularly when $s=s_0$, we have

$$\mathscr{L}_n\left(\delta_n^{-1}A^*_{n,s_0}\right) = T_n, \qquad \text{in } B_R, \tag{4.24}$$

where

$$\mathscr{L}_n = \Delta + (g_n^{ij}-\delta^{ij})\times\nabla^2$$

and $T_n = I + II + III + IV + V$ with

$$I := A^*_{n,s_0} \times \nabla \left(\delta_n^{-1} A^*_{n,s_0}\right), \qquad II := A^*_{n,s_0} \times \delta_n^{-1} F_{A^*_{n,s_0}},$$

$$III := \delta_n \, \phi^*_{n,s_0} \times D_{A^*_{n,s_0}} \phi^*_{n,s_0}, \qquad IV := \delta_n^{-1} \, \partial_s A^*_n|_{s=s_0}, \qquad V := F_{A^*_{n,s_0}} \times \mathcal{F}\left(g, \nabla g\right)|_{x_0+\delta_n y}.$$

In the definition of V, $\mathcal{F}$ is a smooth function.

3. Convergence of connections in good gauge.

Fix k and $l > k$. In this part, we denote by $A^*_{n,l}$ the A^*_{n,s_0} in part 2 with $R = R_l$. Set

$$A_{n,l;k} = \delta_n^{-1} A^*_{n,l} - \left(\delta_n^{-1} A^*_{n,l}\right)_k,$$

where for function S, $(S)_k$ is the average value of S over B_k. Fix $q \in (1,2)$. By Hölder's inequality, one has

$$\left\|\nabla \left(\delta_n^{-1} A^*_{n,l}\right)\right\|_{q;R_k} \le C_{q,R_k} \|\nabla \left(\delta_n^{-1} A^*_{n,l}\right)\|_{2;R_k} \le C_{q,R_k} \|\nabla \left(\delta_n^{-1} A^*_{n,l}\right)\|_{2;R_l}.$$

Notice (4.21)-(4.22). We have

$$\left\|\nabla A^*_{n,l}\right\|_{2;R_l} + \left\|F_{A_{n,s_0}}\right\|_{2;R_l} \le C_{E_0}\, \delta_n. \tag{4.25}$$

Therefore, the above two inequalities imply that

$$\left\|\nabla \left(\delta_n^{-1} A^*_{n,l}\right)\right\|_{q;R_k} \le C_{q,R_k,E_0}; \tag{4.26}$$

Now we estimate I - V in (4.24) with $R = R_l$. By Hölder's inequality and (4.25), one can imply that

$$\| I + II \|_{q;R_k} \le C_{E_0} \, \left\| A^*_{n,l} \right\|_{2q/(2-q);R_l}.$$

In light of Sobolev embedding and (4.21)-(4.22), we have

$$\| I + II \|_{q;R_k} \le C_{q,E_0,R_l}\, \delta_n. \tag{4.27}$$

As for III and IV, one just needs (4.15) and the finite-energy condition so that

$$\| III + IV \|^2_{2;R_k} \le C_{E_0}\, \delta_n^2 + \int_{B_{r_0 \delta_n^{-1}}} \delta_n^{-2} \, \left|\partial_s A_n\right|^2 (\cdot, s_0) \longrightarrow 0, \quad \text{as } n \to \infty. \tag{4.28}$$

The estimate for V is simple. By (4.21), we have

$$\| V \|_{2;R_k} \le C_{E_0}\, \delta_n. \tag{4.29}$$

Notice (4.13). When n is large enough, $\mathscr{L}_n$ is a small perturbation of the Laplace operator Δ. One then can go through the proof of Theorem 9.11 in [4] to obtain a $W^{2,q}$-estimate for $A_{n,l;k}$. More precisely, one can imply that when n is suitably large,

$$\| A_{n,l;k} \|_{2,q;R_k/2} \le C_{q,R_k} \left(\| A_{n,l;k} \|_{q;R_k} + \| T_n \|_{q;R_k} \right).$$

Apply Poincaré inequality. One has

$$\| A_{n,l;k} \|_{2,q;R_k/2} \le C_{q,R_k} \left(\| \nabla (\delta_n^{-1} A^*_{n,l}) \|_{q;R_k} + \| T_n \|_{q;R_k} \right).$$

Notice (4.26)-(4.29) and the compactness of the Sobolev embedding

$$W^{2,q}(B_{R_k/2}) \hookrightarrow W^{1,2}(B_{R_k/2}).$$

We can extract a subsequence by diagonal process, still denoted by $\{n\}$, such that as $n \to \infty$,

$$A_{n,l;k} \longrightarrow A_{l;k}, \qquad \text{weakly in } W^{2,q}(B_{R_k/2}) \text{ and strongly in } W^{1,2}(B_{R_k/2}), \ \forall\, l > k. \tag{4.30}$$

By the lower semi-continuity of $W^{2,q}$-norm, we have

$$\| A_{l;k} \|_{2,q;R_k/2} \le C_{q,R_k,E_0}.$$

Note that the upper bound on the right-hand side of the above inequality is independent of l. Hence, we can keep extracting a subsequence, still denoted by $\{l\}$, such that as $l \to \infty$,

$$A_{l;k} \longrightarrow A_k^*, \qquad \text{weakly in } W^{2,q}(B_{R_k/2}) \text{ and strongly in } W^{1,2}(B_{R_k/2}), \ \forall\, k \in \mathbb{N}. \tag{4.31}$$

4. The Limiting Connection.

In the following, we show that $\{A_k^*\}$ in (4.31) induce a L^2_{loc}-connection on $\mathbb{R}^2$. Firstly, we show that

Lemma 4.3. *If $k_1 < k_2$, then $\nabla A_{k_1}^*$ and $\nabla A_{k_2}^*$ are identical on $B_{R_{k_1}/2}$. Hence,*

$$\eta := \nabla A_k^*, \qquad \text{in } B_{R_k/2}, \ \forall\, k \in \mathbb{N}$$

is well-defined on $\mathbb{R}^2$. Moreover, η is L^2-integrable.

Proof. Note that for any $n, l \in \mathbb{N}$ with $l > k_2$, $A_{n,l;k_1}$ differs from $A_{n,l;k_2}$ by a constant on B_{k_1}. Therefore, when n and l are sufficiently large,

$$\nabla A_{n,l;k_1} \equiv \nabla A_{n,l;k_2}, \qquad \text{in } B_{R_{k_1}/2}.$$

By (4.30)-(4.31), when one sends $n \to \infty$ and $l \to \infty$ successively, then

$$\nabla A_{k_1}^* \equiv \nabla A_{k_2}^*, \qquad \text{on } B_{R_{k_1}/2}.$$

As for the L^2-integrability of η, one can see from (4.25) that

$$\| \nabla A_{n,l;k} \|_{2;R_k/2} \le \left\| \nabla \left(\delta_n^{-1} A_{n,l}^* \right) \right\|_{2;R_l} \le C_{E_0}.$$

Therefore, if we send $n \to \infty$ and $l \to \infty$ successively, then

$$\| \eta \|_{2;R_k/2} = \| \nabla A_k^* \|_{2;R_k/2} \le C_{E_0}, \qquad \forall\, k \in \mathbb{N}.$$

Let $k \to \infty$. We get the desired L^2-integrability of η. □

By Lemma 4.3, we can define a global connection A^* by $\{A_k^*\}$. In fact, one may define

$$A^* = A_1^*, \qquad \text{on } B_{R_1/2}.$$

Note that on $B_{R_1/2}$, A_2^* differs from A_1^* by a constant C_1. Hence, we can define

$$A^* = A_2^* + C_1, \qquad \text{on } B_{R_2/2}$$

so that the definition of A^* can be extended from $B_{R_1/2}$ to $B_{R_2/2}$. By induction, we can define A^* over $\mathbb{R}^2$. Furthermore, one can show from (4.13), (4.24), (4.27)-(4.29) that

Lemma 4.4. *The connection A^* satisfies the harmonic equation*

$$\Delta A^* = 0, \qquad\qquad in\ \mathbb{R}^2.$$

Since $A_{n,l;k}$ is in the Coulomb gauge, A^* satisfies the Coulomb gauge condition $\mathrm{d}^* A^* = 0$ in $\mathbb{R}^2$ as well. Moreover, by the definition of A^* above, we know that η in Lemma 4.3 is the derivative of A^*. That is

$$\eta = \nabla A^*.$$

5. The limiting energy of the rescaled connection

Take a subsequence as in (4.30)-(4.31). Fix an arbitrary $R > 0$. When n and l are large enough,

$$\int_{B_R} \delta_n^{-2} \left| F_{A_{n,s_0}} \right|^2 \mathrm{d}v_{g_n} = \int_{B_R} \delta_n^{-2} \left| F_{A^*_{n,l}} \right|^2 \mathrm{d}v_{g_n}. \tag{4.32}$$

Choose k large enough so that $R_k/2 > R$. Then

$$\nabla \left(\delta_n^{-1} A^*_{n,l} \right) = \nabla A_{n,l;k}, \qquad \text{on } B_R.$$

Hence, if we send $n \to \infty$ and $l \to \infty$ successively, then

$$\nabla \left(\delta_n^{-1} A^*_{n,l} \right) \longrightarrow \nabla A^*, \qquad \text{strongly in } L^2(B_R). \tag{4.33}$$

By (4.21)-(4.22), when n and l are large enough, we have

$$\left\| \delta_n^{-1} A^*_{n,l} \right\|_{1,2;R} \le C_{E_0, R_l}. \tag{4.34}$$

By Sobolev embedding,

$$\left\| \delta_n^{-1} A^*_{n,l} \right\|_{4;R} \le C_{E_0, R_l}. \tag{4.35}$$

Therefore, (4,33) and (4.35) imply that

Lemma 4.5. *Suppose that the limiting connection A^* is represented by*

$$A^* = \sum_l A^*_l\, g_l,$$

where $\{g_l\}$ is an orthonormal basis of the Lie algebra $\mathbf{g}$. Take a subsequence as in (4.30)-(4.31). Then

$$\lim_{n\to\infty} \int_{B_R} \delta_n^{-2} \left| F_{A_{n,s_0}} \right|^2 \mathrm{d}v_{g_n} = \sum_l \int_{B_R} |\nabla \times A^*_l|^2 \,\mathrm{d}x, \qquad \forall\ R > 0.$$

Furthermore, send $R \to \infty$, we have

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_{B_R} \delta_n^{-2} \left| F_{A_{n,s_0}} \right|^2 \mathrm{d}v_{g_n} = \sum_l \int_{\mathbb{R}^2} |\nabla \times A^*_l|^2 \,\mathrm{d}x.$$

Based on all the arguments above, in the end, we show that

Proposition 4.6. *Take a subsequence as in (4.30)-(4.31). Then*

$$\lim_{n\to\infty} \int_{B_R} \delta_n^{-2} \left| F_{A_{n,s_0}} \right|^2 \mathrm{d}v_{g_n} = 0, \qquad \forall\ R > 0.$$

Proof. As discussed in Lemma 4.4, we know that A_l^* satisfies $\mathrm{d}^* A_l^* = 0$ in $\mathbb{R}^2$. Hence, $A_l^* = \nabla^\perp \varphi$, where φ is a scalar function. Still in Lemma 4.4, we know that $\Delta A_l^* \equiv 0$ in $\mathbb{R}^2$. Therefore, one can show that $\Delta\varphi \equiv c$ in $\mathbb{R}^2$, where c is a constant. Notice that

$$\int_{\mathbb{R}^2} |\nabla \times A_l^*|^2 \, \mathrm{d}x = \int_{\mathbb{R}^2} |\Delta\varphi|^2 \, \mathrm{d}x < \infty.$$

This implies that $c = 0$. The proof is then completed by Lemma 4.5. □

6. The limiting energy of the rescaled covariant derivatives.

The convergence of the second term in $E_{n,k}$ (see the end of part 1) is quite similar to the study of Palais-Smale sequences for harmonic map energy. In fact, Fix k and $l > k$. We define $\phi^*_{n,l}$ to be the ϕ^*_{n,s_0} in part 2 with $R = R_l$. Hence,

$$\int_{B_{R_k}} \left| D_{A_{n,s_0}} \phi_{n,s_0} \right|^2 \, \mathrm{d}v_{g_n} = \int_{B_{R_k}} \left| D_{A^*_{n,l}} \phi^*_{n,l} \right|^2 \, \mathrm{d}v_{g_n},$$

where $A^*_{n,l}$ is defined at the beginning of part 3. By the second equation in (1.3), $(A^*_{n,l}, \phi^*_{n,l})$ satisfies

$$-D^*_{A^*_{n,l}} D_{A^*_{n,l}} \phi^*_{n,l} = \sigma_n \cdot \partial_s \phi_n \,|_{s=s_0} - \left(D_{A^*_{n,l}} \nu_i(\phi^*_{n,l}), D_{A^*_{n,l}} \phi^*_{n,l} \right) \nu_i(\phi^*_{n,l}), \qquad \text{in } B_{R_k}, \tag{4.36}$$

where the metric in the above equation is g_n and σ_n is defined in part 2 with $R = R_l$. Notice (4.15) and (4.34). One can apply the similar arguments in [12] to the equation (4.36). Finally, we conclude that

Lemma 4.7. *There exist finitely many non-trivial harmonic maps*

$$\phi^*_s : \mathbb{R}^2 \longmapsto \mathscr{N}, \qquad s = 1, ..., S,$$

such that up to a subsequence, still denoted by $\{n\}$*,*

$$\lim_{n\to\infty} \int_{B_k} \left| D_{A_{n,s_0}} \phi_{n,s_0} \right|^2 \mathrm{d}v_{g_n} = \int_{B_k} |\nabla \phi^*_1|^2 \, \mathrm{d}x + \sum_{s=2}^{L} \int_{\mathbb{R}^2} |\nabla \phi^*_s|^2 \, \mathrm{d}x, \qquad \forall\, k \in \mathbb{N}.$$

If $S = 1$*, then we just have the first term on the right-hand side of the above equality.*

7. Completion of the Proof for Theorem 1.3.

By (4.16)-(4.20) in part 1, Proposition 4.6 and Lemma 4.7, the energy identity (1.4) in Theorem 1.3 holds.

V. Asymptotic Behavior

In this section, we assume that the gradient flow (1.3) admits a global smooth solution on $[0,\infty)$ and study its asymptotic behavior as $t \uparrow \infty$. By Proposition 2.7, we know that

$$\int_0^\infty \int_{\mathscr{M}} |\,\partial_t A\,|^2 + |\,\partial_t \phi\,|^2 \,\mathrm{d}v_g\,\mathrm{d}t < \infty.$$

So we can choose a sequence $t_n \uparrow \infty$ such that

$$\int_{t_n-1}^{t_n+1} \int_{\mathscr{M}} |\,\partial_t A\,|^2 + |\,\partial_t \phi\,|^2 \,\mathrm{d}v_g\,\mathrm{d}t \longrightarrow 0, \qquad \text{as } n \to \infty. \tag{5.1}$$

Meanwhile,

$$\partial_t A(\cdot, t_n),\ \ \partial_t \phi(\cdot, t_n) \longrightarrow 0, \qquad \text{strongly in } L^2(\mathscr{M}), \quad \text{as } n \to \infty. \tag{5.2}$$

Recall the covering $\{\mathscr{U}_\alpha\}$ and the transition functions $\{g_{\alpha,\beta}\}$ that we use in the definition of the prinicpal $\mathscr{G}$-bundle $\mathscr{P}$. For any $x_0 \in \mathscr{M}$, there is a neighborhood $\mathscr{U}_{\alpha(x_0)}$ such that $x_0 \in \mathscr{U}_{\alpha(x_0)}$. Choose r_0 small enough. We have

$$B_{x_0} \subset\subset \mathscr{U}_{\alpha(x_0)},$$

where B_{x_0} is the ball with center x_0 and radius r_0. Suppose that $A_{\alpha(x_0)}(t_n)$ is a local representation of $A(t_n)$ on $\mathscr{U}_{\alpha(x_0)}$. By Uhlenbeck's theorem, there are a Coulomb connection A^*_{n,x_0} and a gauge transformation σ_{n,x_0} such that

$$A^*_{n,x_0} = \sigma_{n,x_0} \cdot A_{\alpha(x_0)}(t_n), \qquad \text{on } B_{x_0}.$$

Meanwhile,

$$\left\| A^*_{n,x_0} \right\|_{1,2;B_{x_0}} \le C\, \| F_{A(t_n)} \|_{2;B_{x_0}}, \tag{5.3}$$

where $C > 0$ is a constant. Since the initial energy of the gradient flow (1.3) is finite, r_0 can be chosen independently of n.

Apply the first equation in (1.3). We know that

$$-D^*_{A^*_{n,x_0}} F_{A^*_{n,x_0}} = G(A,\phi) := \mathrm{Ad}_{\sigma_{n,x_0}} \left(\partial_t A \,|_{t_n} \right) + \left(g_l \phi(t_n), D_{A(t_n)} \phi(t_n) \right) \mathrm{Ad}_{\sigma_{n,x_0}} (g_l)\,. \tag{5.4}$$

Moreover, one can rewrite the left-hand side of (5.4) as follows:

$$-g^{lk} \partial_l \left(\partial_j A^*_{n,x_0;k} - \partial_k A^*_{n,x_0;j} \right) - \nabla A^*_{n,x_0} \times A^*_{n,x_0} - \mathcal{F}(g, \nabla g) \times F_{A^*_{n,x_0}} - A^*_{n,x_0} \times F_{A^*_{n,x_0}},$$

where $\mathcal{F}$ is a smooth function. If we choose normal coordinates in B_{x_0} such that

$$(g_{ij}(x_0)) = (\delta_{ij}),$$

then (5.4) can be rewritten as

$$\Delta A^*_{n,x_0} + \left(g^{lk} - \delta^{lk} \right) \times \nabla^2 A^*_{n,x_0} = G(A,\phi) + \nabla A^*_{n,x_0} \times A^*_{n,x_0} + \mathcal{F}(g, \nabla g) \times F_{A^*_{n,x_0}} + A^*_{n,x_0} \times F_{A^*_{n,x_0}}. \tag{5.5}$$

Note that when r_0 is small enough, the left-hand side of (5.5) is a small perturbation of

$$\Delta A^*_{n,x_0}.$$

Therefore, by (5.3) and the similar arguments as in the proof of Theorem 9.11 in [4], we have

$$\left\| A^*_{n,x_0} \right\|_{2,3/2;\ B_{x_0}/2} \le C_{r_0,E_0} + C_{r_0} \left\| \partial_t A \,|_{t_n} \right\|_{2;B_{x_0}}, \tag{5.6}$$

where $B_{x_0}/\,2$ is the ball with center x_0 and radius $r_0/2$. Note that

$$\{ B_{x_0}/\,2 : x_0 \in \mathscr{M} \}$$

forms a covering of $\mathscr{M}$. By the compactness of $\mathscr{M}$, we can find finitely many balls, denoted by

$$\mathscr{C}^* = \{ B^*_i : B^*_i = B_{x_i}/\,2 \}, \tag{5.7}$$

such that $\mathscr{C}^*$ is a finite covering of $\mathscr{M}$. Notice (5.6) and the compactness of the Sobolev embedding

$$W^{2,3/2}\left(B^*_i \right) \hookrightarrow W^{1,2}\left(B^*_i \right).$$

We can extract a subsequence, still denoted by $\{n\}$, such that for all i,

$$A^*_{n,x_i} \longrightarrow A^*_{x_i}, \qquad \text{weakly in } W^{2,3/2}\left(B^*_i \right) \text{ and strongly in } W^{1,2}(B^*_i), \quad \text{as } n \to \infty. \tag{5.8}$$

Define A^*_n and A^* so that for all i,

$$A^*_n|_{B^*_i} = A^*_{n,x_i} \qquad \text{and} \qquad A^*|_{B^*_i} = A^*_{x_i}.$$

Note that on $B^*_i \cap B^*_j$,

$$A^*_{n,x_i} = \sigma_{n,x_i} \cdot A_{\alpha(x_i)}(t_n) = \sigma_{n,x_i} \cdot g_{\alpha(x_i),\alpha(x_j)} \cdot A_{\alpha(x_j)}(t_n) = \sigma_{n,x_i} \cdot g_{\alpha(x_i),\alpha(x_j)} \cdot \sigma^{-1}_{n,x_j} \cdot A^*_{n,x_j}. \tag{5.9}$$

Then A^*_n is a connection 1-form on $\mathscr{P}_n$. Here $\mathscr{P}_n$ is a principal $\mathscr{G}$-bundle over $\mathscr{M}$ determined by $\{B^*_i\}$ and the transition functions

$$\{ g_{i,j;n} \} = \left\{ \sigma_{n,x_i} \cdot g_{\alpha(x_i),\alpha(x_j)} \cdot \sigma^{-1}_{n,x_j} \right\}.$$

Moreover, by (5.9), one has

$$\mathrm{d}g_{i,j;n} = g_{i,j;n} A^*_{n,x_j} - A^*_{n,x_i} g_{i,j;n}, \qquad \text{on } B^*_i \cap B^*_j. \tag{5.10}$$

Apply the compactness of Sobolev embedding. Up to a subsequence, we have for all i, j and $p > 2$,

$$g_{i,j;n} \longrightarrow g^*_{i,j}, \qquad \text{weakly in } W^{2,2}\left(B^*_i \cap B^*_j \right) \text{ and strongly in } W^{1,p}\left(B^*_i \cap B^*_j \right).$$

Clearly, $\{B^*_i\}$ and $\left\{ g^*_{i,j} \right\}$ determine a new principal $\mathscr{G}$-bundle, denoted by $\mathscr{P}^*$, over $\mathscr{M}$. One can check that A^* is a connection 1-form on $\mathscr{P}^*$.

Now we study the convergence of sections. Similarly as before, we define ϕ^*_n so that for all i,

$$\phi^*_n|_{B^*_i} = \phi^*_{n,x_i},$$

where

$$\phi^*_{n,x_i} := \sigma_{n,x_i} \cdot \phi_{\alpha(x_i)}(t_n)$$

with $\phi_{\alpha(x_i)}(t_n)$ a local representation of $\phi(t_n)$ on $\mathscr{U}_{\alpha(x_i)}$. Obviously, ϕ^*_n is a section of the fibre bundle

$$\mathscr{E}_n := \mathscr{P}_n \times_{\mathscr{G}} \mathscr{N}.$$

Let

$$\Sigma = \left\{ y_0 \in \mathscr{M} : \lim_{r\to 0} \limsup_{n\to\infty} \int_{B_r(y_0)} e(A,\phi)\,\mathrm{d}v_g \Big|_{t=t_n} \geq \epsilon_1 = \epsilon_0/2 \right\},$$

where ϵ_0 and ϵ_1 are the same as in Proposition 3.4. By similar arguments as in the proof of part (2) of Remark 3.5, Σ is a finite subset of $\mathscr{M}$. Choose a sequence $r_k \downarrow 0$. We can define a r_k-neighborhood of Σ as follows:

$$\Sigma_k := \bigcup_{y\in\Sigma} B_{r_k}(y).$$

Fix k. For any $x_1 \in \Sigma_k^c$, where Σ_k^c is the complement set of Σ_k in $\mathscr{M}$, one can find some B_i^* such that $x_1 \in B_i^*$ and ϕ_n^* has a local representation ϕ_{n,x_i}^* in B_i^*. Choose r_1 small enough so that $B_{r_1}(x_1)$ is contained in B_i^*. Note that one may have more than one balls in $\mathscr{C}^*$ (see (5.7)) which contain x_1. In this case, we choose r_1 small enough such that $B_{r_1}(x_1)$ is contained in the intersection of these balls which contain x_1. Moreover, we can require that

$$\overline{B}_{r_1}(x_1) \bigcap \Sigma = \emptyset.$$

Since $x_1 \in \Sigma_k^c$, we can keep choosing r_1 small enough so that

$$\int_{B_{r_1}(x_1)} e(A,\phi)\,\mathrm{d}v_g \Big|_{t_n} < \epsilon_1, \qquad \text{for } n \text{ large.}$$

By (5.1) and (3.2), one may imply that for some $r_2 < r_1$, where r_2 depends on x_1, ϵ_0, E_0 and r_1, we have

$$\sup_{t\in[t_n - r_2^2, t_n]} \int_{B_{r_2}(x_1)} e(A,\phi)\,\mathrm{d}v_g \Big|_{t} < \epsilon_0, \qquad \text{for } n \text{ large.}$$

By ϵ-regularity in Proposition 3.3, we know that

$$\sup_{B_{r_2/3}(x_1)} e(A(t_n),\phi(t_n)) \leq C\, r_2^{-2}. \tag{5.11}$$

Therefore, in light of (5.2), (5.11) and the second equation in (1.3), the $W^{2,2}$-norm of ϕ_{n,x_i}^* in $B_{r_2/6}(x_1)$ is uniformly bounded. So are other representations of ϕ_n^*. Clearly

$$\left\{ B_{r_2/6}(x_1) : x_1 \in \Sigma_k^c \right\}$$

forms a covering of Σ_k^c. By the compactness, we can extract a finite covering of Σ_k^c, denoted by

$$\mathscr{C}_k = \left\{ B_{r_{k,s}}(x_{k,s}) \right\},$$

so that all representations of ϕ_n^* are uniformly bounded in $W^{2,2}$-norm on $B_{r_{k,s}}(x_{k,s})$. Furthermore,

$$\mathscr{C} = \bigcup_k \mathscr{C}_k$$

forms a countable covering of $\mathscr{M} \setminus \Sigma$. Fix a $p > 2$. By the compactness of the Sobolev embedding

$$W^{2,2}(B) \hookrightarrow W^{1,p}(B),$$

where B is a ball and the diagonal process, we can extract a subsequence, still denoted by $\{n\}$, so that

$$\phi_n^* \longrightarrow \phi^*, \qquad \text{strongly in } W^{1,p}\left(B_{r_{k,s}}(x_{k,s})\right), \quad \text{for all } k \text{ and } s. \tag{5.12}$$

Here, (5.12) should be understood as the convergence for all representations of ϕ_n^*.

Based on all the arguments above, for any $x_0 \in \mathscr{M} \setminus \Sigma$, we can find a ball

$$B = B_{r_{k,s}}(x_{k,s})$$

so that $x_0 \in B$. By (1.3), (5.2), (5.8) and (5.12), if we send $n \to \infty$, then (A^*, ϕ^*) solves (1.2) on B weakly. Since on B, all representations of A^* satisfy the Coulomb gauge condition and all representations of ϕ^* are $W^{1,p}$-regular for some $p > 2$. Then standard elliptic estimates imply that (A^*, ϕ^*) is a smooth solution of (1.2) on B. Moreover, (A^*, ϕ^*) must be a smooth solution of (1.2) away from the points in Σ in that x_0 is arbitrary. In Section VI, we remove the singularities and show that (A^*, ϕ^*) is indeed a global smooth solution of (1.2) over $\mathscr{M}$. Therefore, take $n \to \infty$ in (5.10). We have

$$\mathrm{d}g_{i,j}^* = g_{i,j}^* A_{x_j}^* - A_{x_i}^* g_{i,j}^*,$$

which implies that the limiting principal $\mathscr{G}$-bundle $\mathscr{P}^*$ is a smooth principal $\mathscr{G}$-bundle.

VI. Removability of Singularites

In this section, by following the arguments in [14], we study the removability of singularities for the model of gauged harmonic maps. Since the theory is local, for our convenience, we can assume that the Riemannian metric is the usual Euclidean metric.

1. Some Assumptions.

For some $\delta_* > 0$ sufficiently small, we suppose that (A, ϕ) satisfies the assumptions shown as follows:

(A1). For the ball B_{δ_*} with center 0 and radius δ_*, (A, ϕ) solves (1.2) smoothly on $B_{\delta_*} \setminus \{0\}$;

(A2). On B_{δ_*}, A is in the Coulomb gauge and is a $W^{2,3/2}$-strong solution of the first equation in (1.2);

(A3). The energy of (A, ϕ) in B_{δ_*} is finite and small enough.

All these assumptions can be naturally satisfied by the arguments in Section V. For example, (A2) can be obtained from (5.8). With these assumptions, one may derive some quick results. We list these results in the following for our future use. By (A2)-(A3), one can find a positive constant C independent of δ so that

$$\int_{B_\delta} |A|^2 + |\nabla A|^2 + |\nabla \phi|^2 \le C, \qquad \forall\ \delta \le \delta_*. \tag{6.1}$$

From (A2) and Morrey's inequality, we have

$$\|A\|_{C^{0,2/3}(B_\delta)} \le C_\delta\, \|A\|_{2,3/2;\,\delta} \le C, \qquad \forall\ \delta \le \delta_*. \tag{6.2}$$

Still by (A2), we can apply Sobolev embedding and imply that

$$\|\nabla A\|_{6;\delta} \le C_\delta\, \|A\|_{2,3/2;\,\delta} \le C, \qquad \forall\ \delta \le \delta_*. \tag{6.3}$$

2. Refined estimate for $e(A,\phi)$

Fix an arbitrary $\delta < \delta_*$. We consider the pointwise estimate of $e(A,\phi)$ on $B_\delta \setminus \{0\}$. Note that in the following, $C > 0$ is a constant independent of δ. Choose an arbitrary $x_0 \in B_\delta \setminus \{0\}$. Since (A,ϕ) is a stationary solution of (1.3) on $B_{|x_0|/2}(x_0)$, therefore by (A3) and Proposition 3.3, we have

$$\sup_{B_{\rho_0}(x_0)} e(A,\phi) \le C\,|x_0|^{-2}, \tag{6.4}$$

where $\rho_0 := |x_0|/6$. Rescale (A,ϕ) within $B_{\rho_0}(x_0)$ by setting

$$a = \rho_0 A(x_0 + \rho_0\, y), \qquad s = \phi(x_0 + \rho_0\, y), \qquad \forall\ y \in B_1.$$

Then by Bochner-type inequality in Proposition 3.2, we have

$$-\Delta h \le C\,(1 + |F_a|\,)\,h + \left(D_a \nu_i(s), D_a s\right)^2,$$

where $h = 1/2\left(\rho_0^{-2}\,|F_a|^2 + |D_a s|^2\right)$ is the rescaled energy for (a,s). By (6.4), one has

$$\sup_{B_1} h \le C.$$

Particularly, we have

$$\sup_{B_1} |F_a|^2 \le C\,\rho_0^2.$$

If ρ_0 is small enough (necessarily if δ_* is small enough), one can choose good gauge for a in B_1 to show that the gauge invariant quantity $|D_a \nu_i(s)|$ is uniformly bounded in B_1. Therefore, one can imply that

$$-\Delta h \le C\,h, \qquad \text{in } B_1.$$

Apply Harnack's inequality, we know that

$$\sup_{B_{1/2}} h \le C \int_{B_1} h = C \int_{B_{\rho_0}(x_0)} e(A,\phi) \le C \int_{B_{2|x_0|}} e(A,\phi).$$

Hence, we have the following refined estimate for $e(A,\phi)$:

Lemma 6.1. *There is a large constant $C > 0$ such that for all $x_0 \in B_\delta \setminus \{0\}$, we have*

$$e(A,\phi)(x_0) \le C\,|x_0|^{-2} \int_{B_{2|x_0|}} e(A,\phi).$$

3. Energy Stress Tensor

We define the energy stress tensor as follows:

$$T = |F_{12}|^2 I_2 + \begin{pmatrix} |\nabla_{A,1}\phi|^2 - |\nabla_{A,2}\phi|^2, & 2\,\nabla_{A,1}\phi \cdot \nabla_{A,2}\phi \\ 2\,\nabla_{A,1}\phi \cdot \nabla_{A,2}\phi, & |\nabla_{A,2}\phi|^2 - |\nabla_{A,1}\phi|^2 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix. If (A,ϕ) solves (1.2) on $B_\delta \setminus \{0\}$, then

$$\partial_j T_{kj} = 0, \qquad \text{on } B_\delta \setminus \{0\},\ \ k = 1, 2. \tag{6.5}$$

Set $z = x_1 + i\, x_2$ and $\omega = T_{11} - i\, T_{12}$, where $i^2 = -1$. One can show that the imaginary part of $\omega\, z\, \mathrm{d}z$ is

$$\Im(\omega\, z\, \mathrm{d}z) = (x_2 T_{11} - x_1 T_{12})\, \mathrm{d}x_1 + (x_1 T_{11} + x_2 T_{12})\, \mathrm{d}x_2.$$

We calculate its integration over ∂B_r ($r < \delta$). As a convention, for $s < r$, we denote by $A(s, r)$ the annulus

$$\left\{ x \in \mathbb{R}^2 : s \le |x| \le r \right\}.$$

Apply Stokes' theorem and (6.5). We have

$$\int_{\partial A(s,r)} \Im(\omega\, z\, \mathrm{d}z) = \int_{A(s,r)} x_2 \left(\partial_1 T_{12} - \partial_2 T_{11} \right) = 4 \int_{A(s,r)} x_2\, \nabla_{A,1} \phi \cdot F_{12} \phi.$$

Let $s \to 0$, the right-hand side of the above equality converges to the integration over B_r. As for the most left-hand side of the above equality, we have

$$\left| \int_{\partial B_s} \Im\left(\omega\, z\, \mathrm{d}z \right) \right| = \left| s^2 \int_0^{2\pi} T_{11} \cos 2\theta + T_{12} \sin 2\theta\, \mathrm{d}\theta \right| \le C\, s^2 \int_{\partial B_s} e(A, \phi)\, \mathrm{d}\theta.$$

Apply Lemma 6.1. It can be shown that

$$\left| \int_{\partial B_s} \Im\left(\omega\, z\, \mathrm{d}z \right) \right| \le C \int_{B_{2s}} e(A, \phi) \longrightarrow 0, \qquad \text{as } s \to 0.$$

Therefore, we imply that

$$\Re \int_{\partial B_r} \omega z^2\, \mathrm{d}\theta = \int_{\partial B_r} \Im(\omega\, z\, \mathrm{d}z) = 4 \int_{B_r} x_2\, \nabla_{A,1} \phi \cdot F_{12} \phi \le C\, r \int_{B_r} e(A, \phi). \tag{6.6}$$

By polar coordinate, one can show that the most left-hand side of (6.6) is bounded from below by

$$\int_{\partial B_r} \left(r^2\, |\, \phi_r\, |^2 - |\, \phi_\theta\, |^2 \right)\, \mathrm{d}\theta - \epsilon^*\, r^2 \int_{\partial B_r} |\, \nabla \phi\, |^2\, \mathrm{d}\theta - C_{\epsilon^*}\, r^2 \int_{\partial B_r} |F_A|^2 + |\, A\, |^2\, \mathrm{d}\theta,$$

where $\epsilon^* > 0$ is a constant which can be chosen arbitrarily small. Therefore, we have

Lemma 6.2. *If (A, ϕ) satisfies (1.2) in $B_\delta \setminus \{0\}$, then for all $r \in (0, \delta)$,*

$$\int_{\partial B_r} \left(r^2\, |\, \phi_r\, |^2 - |\, \phi_\theta\, |^2 \right)\, \mathrm{d}\theta \le C\, r \int_{B_r} e(A, \phi) + \epsilon^*\, r^2 \int_{\partial B_r} |\, \nabla \phi\, |^2\, \mathrm{d}\theta + C_{\epsilon^*}\, r^2 \int_{\partial B_r} |F_A|^2 + |A|^2\, \mathrm{d}\theta,$$

where $C > 0$ is a constant independent of r and (A, ϕ). $C_{\epsilon^*} > 0$ is a constant depending on ϵ^*.

4. Removability of Singularites

As a convention, for $m \in \mathbb{N}$, $A_{m,\delta}$ denotes the annulus $A\left(2^{-m}\delta, 2^{-m+1}\delta \right)$. In the following, q is a function defined on $B_\delta \setminus \{0\}$ such that on each $A_{m,\delta}$,

$$q = C_{m,1} + C_{m,2} \log |x|,$$

where $C_{m,1}$ and $C_{m,2}$ are constant vectors such that for all $m \in \mathbb{N}$,

$$q \equiv \fint_{\partial B_{2^{-m+1}\delta}} \phi\, \mathrm{d}\theta, \qquad \text{on } \partial B_{2^{-m+1}\delta}. \tag{6.7}$$

We now compare q with ϕ. Note that for all $r \in \big(2^{-m}\delta, 2^{-m+1}\delta\big)$,

$$|\, q(r) - \phi(r,\theta)\,| \le \big|\, q(r) - q(2^{-m+1}\delta)\,\big| + \big|\, q(2^{-m+1}\delta) - \phi(r,\theta)\,\big|\,.$$

Hence, apply the maximum principal on the annulus $A_{m,\delta}$ for the function q, we have

$$|\, q(r) - \phi(r,\theta)\,| \le \big|\, q(2^{-m}\delta) - q(2^{-m+1}\delta)\,\big| + \big|\, q(2^{-m+1}\delta) - \phi(r,\theta)\,\big|\,, \qquad \text{on } A_{m,\delta}.$$

By the definition of q, we know that

$$q(2^{-m}\delta) - q(2^{-m+1}\delta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(2^{-m}\delta,\theta) - \phi(2^{-m+1}\delta,\theta)\,\mathrm{d}\theta$$

and

$$q(2^{-m+1}\delta) - \phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(2^{-m+1}\delta,\alpha) - \phi(r,\theta)\,\mathrm{d}\alpha.$$

Therefore, one can imply that

$$|\, q(r) - \phi(r,\theta)\,| \le 2 \sup_{x,y\in A_{m,\delta}} |\,\phi(x) - \phi(y)\,|\,, \qquad \forall\, r \in \big(2^{-m}\delta, 2^{-m+1}\delta\big)\,. \tag{6.8}$$

Note that

$$|\,\phi(x) - \phi(y)\,| \le 2^{-m+3}\,\delta\,\|\nabla\phi\|_{\infty;A_{m,\delta}}\,, \qquad \text{for all } x,y \in A_{m,\delta}.$$

Therefore, by the boundedness of A in (6.2), we know that

$$|\,\phi(x) - \phi(y)\,| \le 2^{-m+3}\,\delta\,\|D_A\phi\|_{\infty;A_{m,\delta}} + C\,2^{-m}, \qquad \text{for all } x,y \in A_{m,\delta}.$$

Apply Lemma 6.1. $\|D_A\phi\|_{\infty;A_{m,\delta}}$ can be controlled and moreover, one can show that

$$|\,\phi(x) - \phi(y)\,| \le C\left(\int_{B_{2^{-m+2}\delta}} e(A,\phi)\right)^{1/2} + C\,2^{-m}, \qquad \text{for all } x,y \in A_{m,\delta}. \tag{6.9}$$

By (6.8)-(6.9),

$$|\, q(r) - \phi(r,\theta)\,| \le C\left(\int_{B_{2^{-m+2}\delta}} e(A,\phi)\right)^{1/2} + C\,2^{-m}, \qquad \text{on } A_{m,\delta}. \tag{6.10}$$

In the following, we estimate the gradient of $q - \phi$. Note that for any $k \in \{0,1,2,...\}$,

$$\int_{B_{2^{-k}\delta}} |\,\nabla q - \nabla\phi\,|^2 = \sum_{m=k+1}^{\infty}\int_{A_{m,\delta}} |\,\nabla q - \nabla\phi\,|^2\,.$$

Integrate by parts. We know that

$$\int_{B_{2^{-k}\delta}} |\,\nabla q - \nabla\phi\,|^2 = -\int_{B_{2^{-k}\delta}} (q-\phi)\cdot(\Delta q - \Delta\phi) + \sum_{m=k+1}^{\infty}\int_{\partial A_{m,\delta}} (q-\phi)\cdot(q_{\vec{n}} - \phi_{\vec{n}})\,\mathrm{d}s, \tag{6.11}$$

where for fixed $m \in \mathbb{N}$, $q_{\vec{n}}$ and $\phi_{\vec{n}}$ are derivatives of q and ϕ along the outer normal direction of $\partial A_{m,\delta}$. By the boundary condition (6.7) and the fact that q is radial, one can show that

$$\int_{\partial A_{m,\delta}} (q-\phi)\cdot q_{\vec{n}}\,\mathrm{d}s = 0. \tag{6.12}$$

For any fixed $S \in \mathbb{N}$, $S \geq k+1$, after cancellation, we know that

$$\sum_{m=k+1}^{S} \int_{\partial A_{m,\delta}} (q-\phi) \cdot \phi_{\vec{n}} \, \mathrm{d}s = \int_{\partial B_{2^{-k}\delta}} (q-\phi) \cdot \phi_r \, \mathrm{d}s - \int_{\partial B_{2^{-S}\delta}} (q-\phi) \cdot \phi_r \, \mathrm{d}s.$$

By Lemma 6.1, (6.2) and (6.10), one can estimate the last term in the above equality by

$$\left| 2^S \delta^{-1} \int_{\partial B_{2^{-S}\delta}} (q-\phi) \cdot (x_j \partial_j \phi) \, \mathrm{d}s \right| \leq C \left(4^{-S} + \int_{B_{2^{-S+2}\delta}} e(A,\phi) \right) \longrightarrow 0, \qquad \text{as } S \to \infty.$$

Therefore,

$$\sum_{m=k+1}^{\infty} \int_{\partial A_{m,\delta}} (q-\phi) \cdot \phi_{\vec{n}} \, \mathrm{d}s = \int_{\partial B_{2^{-k}\delta}} (q-\phi) \cdot \phi_r \, \mathrm{d}s. \tag{6.13}$$

By (6.11)-(6.13) and the fact that $\Delta q = 0$, one can show that

$$\int_{B_{2^{-k}\delta}} |\nabla q - \nabla \phi|^2 = \int_{B_{2^{-k}\delta}} (q-\phi) \cdot \Delta\phi - \int_{\partial B_{2^{-k}\delta}} (q-\phi) \cdot \phi_r \, \mathrm{d}s. \tag{6.14}$$

We estimate the last term in (6.14). It is bounded by

$$2^{-k}\delta \left(\int_{\partial B_{2^{-k}\delta}} |q-\phi|^2 \, \mathrm{d}\theta \right)^{1/2} \left(\int_{\partial B_{2^{-k}\delta}} |\phi_r|^2 \, \mathrm{d}\theta \right)^{1/2}.$$

Notice (6.7). One can apply Poincaré inequality and imply that

$$\left| \int_{\partial B_{2^{-k}\delta}} (q-\phi) \cdot \phi_r \, \mathrm{d}s \right| \leq C 2^{-k}\delta \left(\int_{\partial B_{2^{-k}\delta}} |\phi_\theta|^2 \, \mathrm{d}\theta \right)^{1/2} \left(\int_{\partial B_{2^{-k}\delta}} |\nabla\phi|^2 \, \mathrm{d}\theta \right)^{1/2}. \tag{6.15}$$

As for the first term on the right-hand side of (6.14), one has

$$\left| \int_{B_{2^{-k}\delta}} (q-\phi) \cdot \Delta\phi \right| \leq \|q-\phi\|_{\infty; B_{2^{-k}\delta}} \int_{B_{2^{-k}\delta}} |\Delta\phi|.$$

In light of (6.10), for any $\epsilon^* > 0$ arbitrarily small, we can find $k_0 = k_0(A, \phi, \epsilon^*)$ large enough such that

$$\|q-\phi\|_{\infty; B_{2^{-k_0}\delta}} < \epsilon^*.$$

Therefore,

$$\left| \int_{B_{2^{-k_0}\delta}} (q-\phi) \cdot \Delta\phi \right| \leq \epsilon^* \int_{B_{2^{-k_0}\delta}} |\Delta\phi|. \tag{6.16}$$

Suppose that $\delta_0 = 2^{-k_0}\delta$. Hence, (6.14)-(6.16) imply that

$$\int_{B_{\delta_0}} |\nabla q - \nabla\phi|^2 \leq \epsilon^* \int_{B_{\delta_0}} |\Delta\phi| + C\,\delta_0 \left(\int_{\partial B_{\delta_0}} |\phi_\theta|^2 \, \mathrm{d}\theta \right)^{1/2} \left(\int_{\partial B_{\delta_0}} |\nabla\phi|^2 \, \mathrm{d}\theta \right)^{1/2}. \tag{6.17}$$

By the second equation in (1.2), we then can show from (6.2) that

$$|\Delta\phi| \leq C \left(|A|^2 + |\nabla\phi|^2 \right) \leq C + C\,|\nabla\phi|^2, \qquad \text{on } B_{\delta_0}.$$

Apply the above inequality in (6.17). We have

$$\int_{B_{\delta_0}} |\nabla q - \nabla \phi|^2 \leq C\,\delta_0^2 + C\,\epsilon^* \int_{B_{\delta_0}} |\nabla \phi|^2 + C\,\delta_0^2 \int_{\partial B_{\delta_0}} |\nabla \phi|^2 \;\mathrm{d}\theta. \tag{6.18}$$

Consider the left-hand side of (6.18). Note that q is a radial function. Therefore,

$$\int_{B_{\delta_0}} |\nabla q - \nabla \phi|^2 \geq \int_{B_{\delta_0}} |x|^{-2}\; |\phi_\theta|^2 . \tag{6.19}$$

By Lemma 6.2, one can show that

$$\int_{B_{\delta_0}} |\phi_r|^2 - |x|^{-2}\,|\phi_\theta|^2 \leq C \int_0^{\delta_0} \int_{B_r} e(A,\phi) + \epsilon^* \int_{B_{\delta_0}} |\nabla \phi|^2 + C_{\epsilon^*} \int_{B_{\delta_0}} |F_A|^2 + |A|^2.$$

Moreover,

$$\int_{B_{\delta_0}} |x|^{-2}\,|\phi_\theta|^2 \geq \left(\frac{1}{2} - \epsilon^*\right) \int_{B_{\delta_0}} |\nabla\phi|^2 - C \int_0^{\delta_0} \int_{B_r} e(A,\phi) - C_{\epsilon^*} \int_{B_{\delta_0}} |F_A|^2 + |A|^2. \tag{6.20}$$

Choose ϵ^* small enough. One then can show, by (6.2)-(6.3), (6.18)-(6.20), that

$$\int_{B_{\delta_0}} |\nabla\phi|^2 \leq C\,\delta_0 + C\,\delta_0^2 \int_{\partial B_{\delta_0}} |\nabla \phi|^2 \;\mathrm{d}\theta.$$

Since $\delta_0 = 2^{-k_0}\,\delta$ and $\delta > 0$ is an arbitrary number less than δ_*, then by the above inequality, we have

$$\int_{B_r} |\nabla\phi|^2 \leq C\,r + C\,r^2 \int_{\partial B_r} |\nabla \phi|^2 \;\mathrm{d}\theta, \qquad\qquad \forall\, r < r_*, \tag{6.21}$$

where $r_* = 2^{-k_0}\delta_*$. Solve (6.21), we get

$$\int_{B_r} |\nabla\phi|^2 \leq C\,r^\alpha, \qquad\qquad \forall\; r < r_*,$$

where $\alpha \in (0,1)$ is a constant. By Lemma 6.1, (6.2)-(6.3), we then have

$$|D_A\phi|^2\,(x_0) \leq C\,|x_0|^{\alpha-2}, \qquad\qquad \text{for all } x_0 \text{ with } \;|x_0| < r_*/2.$$

Hence, $D_A\phi$ is $L^{2\beta}$-integrable in B_{δ_*}, for some $\beta \in (1, 2/(2-\alpha))$. The proof of Theorem 1.5 is then completed by applying the standard elliptic estimates for the equation (1.2).

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