AUG-12 GROUP 4 EXERCISES

Calculus

You are reminded of some facts.

Power rule:	$\frac{d}{dx}(x^n) = nx^{n-1}$
Product rule:	$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
Quotient rule:	$\frac{d}{dx}(f(x)/g(x)) = (g(x)f'(x) - f(x)g'(x))/g(x)^2$
Chain rule:	$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Excercise 0.1. Let $f(x)$ be differentiable everywhere. Let $g(x) = (1+x)f(x)$, $h(x) = f(x)/(1+x)$. Clearly $f(0) = g(0) = h(0)$. This leads us to the following.

(1) Let $g(x) = (1+x^2)f(x)$. Let $h(x) = f(x)/(1+x^2)$. Show that $f'(0) = g'(0) = h'(0)$.
(2) Let $g(x) = (1+x^3)f(x)$. Let $h(x) = f(x)/(1+x^3)$. Show that $f'(0) = g'(0) = h'(0)$. Generalise this further.

Excercise 0.2. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Show that $f'(0) = 0$. Show that $f''(0) = 0$. Show that all derivatives of $f(x)$ at $x = 0$ is 0. (Plot $f(x)$ on desmos if you wish)

Remark. You would expect that having all derivatives of $f(x)$ at $x = 0$ be zero imply that $f(x)$ is zero everywhere, but we have just shown that's not the case!

Excercise 0.3. Suppose $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_mx^m$ for some positive integers $n, m \geq 4$ and real numbers $a_0, \ldots a_n, b_0, \ldots b_m$.

Show that $f(0) = a_0, f'(0) = a_1$ and $f''(0) = \frac{1}{2}a_2$.

Suppose $f(x)g(x) = c_0 + c_1x + c_2x^2 + \ldots$ Show that $c_1 = a_0b_1 + a_1b_0$. Hence verify the product rule for polynomials at $x = 0$.

Let $f(g(x)) = e_0 + e_1x + e_2x^2 + \ldots$. Show that $e_0 = a_0 + a_1b_0 + a_2b_0^2 + \cdots = f(g(0))$ and $e_1 = a_1b_1 + 2a_2b_1 + \ldots$ Hence verify the chain rule for polynomials at $x = 0$.

It is known that if $g(0) \neq 0$, then there exists some interval around $x = 0$ such that $1/g(x) = d_0 + d_1x + d_2x^2 + \ldots$ for some real numbers $d_0, d_1, \ldots$. Show that $d_0 = 1/b_0$ and $d_1 = -b_1/b_0$. Hence verify the quotient rule for $f(x)/g(x)$ at $x = 0$.

[Example: For $x \in (-1, 1)$, we have $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ by sum of geometric series.]

Remark. The point of the above exercise is that if you assume that all nice functions can be written as a power series, then the rules of differentiation are rather natural / can be deduced without limits!

Projectile motion

One application of vector-valued functions is to understand projectile motion.

Excercise 0.4. A cannon at the origin O fires a shell with speed V at an angle α to the horizontal. The shell experiences a constant acceleration vector $(0, -g)$ due to gravity.

(1) Write down Newton's second law for the shell, with suitable initial conditions. Solve the differential equation (by repeated integration) to show that the trajectory of the shell is given by

$$x(t) = tV\cos\alpha, \qquad y(t) = -\frac{1}{2}gt^2 + tV\sin\alpha$$

[Hint: What is $\mathbf{r}''(t)$? What is $\mathbf{r}'(0)$ and $\mathbf{r}(0)$? If you don't know integration right now, can you guess what $\mathbf{r}'(0)$ and $\mathbf{r}(0)$ are?]

(2) Express t in terms of x, α and hence show

$$y = \frac{-g}{2V^2\cos^2\alpha}x^2 + x\tan\alpha.$$

(3) Treating x as constant, show that the derivative of y with respect to α is

$$\frac{\partial y}{\partial \alpha} = \frac{x}{\cos^2\alpha}\left(\frac{-gx\tan\alpha}{V^2} + 1\right)$$

[Hint: Derive $\tan\alpha$ using the quotient rule]

(4) ** Suppose that V is fixed but the angle α can be varied. Show that upper boundary curve $u = u(x)$ of the set of points in the (x, y) plane which it is possible to hit with a shell is

$$u(x) = \frac{-g}{2V^2}x^2 + \frac{V^2}{2g}$$

[Hint: Use part 3, what does $\frac{\partial y}{\partial \alpha} = 0$ signify?]

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GEOMETRY – SHEET 1 – Vector Geometry in $\mathbb{R}^n$

1. (i) Show that the distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are collinear (i.e. lie on a line) in $\mathbb{R}^n$ if and only if the vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are linearly dependent.

(ii) Show that the vectors $\mathbf{u}=(1,2,-3)$ and $\mathbf{v}=(6,3,4)$ are perpendicular in $\mathbb{R}^3$. Verify directly Pythagoras' Theorem for the right-angled triangles with vertices $\mathbf{0},\mathbf{u},\mathbf{v}$ and vertices $\mathbf{0},\mathbf{u},\mathbf{u}+\mathbf{v}$.

(iii) Let $\mathbf{v},\mathbf{w}$ be vectors in $\mathbb{R}^n$. Show that if $\mathbf{v}\cdot\mathbf{x}=\mathbf{w}\cdot\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ then $\mathbf{v}=\mathbf{w}$.

2. Consider the two lines in $\mathbb{R}^3$ given parametrically by

$$\mathbf{r}(\lambda)=(1,3,0)+\lambda(2,3,2), \qquad \mathbf{s}(\mu)=(2,1,0)+\mu(0,2,1).$$

Show that the shortest distance between these lines is $\sqrt{3/7}$ by solving the simultaneous equations

$$(\mathbf{r}(\lambda)-\mathbf{s}(\mu))\cdot(2,3,2)=0, \qquad (\mathbf{r}(\lambda)-\mathbf{s}(\mu))\cdot(0,2,1)=0.$$

What geometry do these equations encode?

Optional – requires knowledge of partial derivatives. The shortest distance could also be found by solving the equations

$$\frac{\partial}{\partial\lambda}\left(|\mathbf{r}(\lambda)-\mathbf{s}(\mu)|^2\right)=0, \qquad \frac{\partial}{\partial\mu}\left(|\mathbf{r}(\lambda)-\mathbf{s}(\mu)|^2\right)=0.$$

Determine these equations and explain why they are (essentially) the same as the previous two.

3. Let $(x,y,z)=(s+t+2,3s-2t+1,4s-3t)$. Show that, as s,t vary, the point (x,y,z) ranges over a plane with equation $ax+by+cz=d$ which you should determine.

4. Determine, in the form $\mathbf{r}\cdot\mathbf{n}=c$, the equations of each of the following planes in $\mathbb{R}^3$;
(i) the plane containing the points $(1,0,0)$, $(1,1,0)$, $(0,1,1)$;
(ii) the plane containing the point $(2,1,0)$ and the line $x=y=z$;
(iii) the two planes containing the points $(1,0,1)$, $(0,1,1)$ and which are tangential to the unit sphere, centre $\mathbf{0}$.

5. Given a vector $\mathbf{a}\in\mathbb{R}^2$ and a constant $0<\lambda<1$, define $\mathbf{b}=\mathbf{a}/\left(1-\lambda^2\right)$ and prove that

$$\frac{|\mathbf{r}-\mathbf{a}|^2-\lambda^2|\mathbf{r}|^2}{1-\lambda^2}=|\mathbf{r}-\mathbf{b}|^2-\lambda^2|\mathbf{b}|^2.$$

Deduce *Apollonius' Theorem* which states that if O and A are fixed points in the plane, then the locus of all points X, such that $|AX|=\lambda\,|OX|$, is a circle. Find its centre and radius.

6. *(Optional)* A tetrahedron $ABCD$ has vertices with respective position vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ from an origin O inside the tetrahedron. The lines AO,BO,CO,DO meet the opposite faces in E,F,G,H.

(i) Show that a point lies in the plane BCD if and only if it has position vector $\lambda\mathbf{b}+\mu\mathbf{c}+\nu\mathbf{d}$ where $\lambda+\mu+\nu=1$.

(ii) There are $\alpha,\beta,\gamma,\delta$, not all zero, such that $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=\mathbf{0}$. Show that E has position vector

$$\frac{-\alpha\mathbf{a}}{\beta+\gamma+\delta}.$$

(iii) Deduce that

$$\frac{|AO|}{|AE|}+\frac{|BO|}{|BF|}+\frac{|CO|}{|CG|}+\frac{|DO|}{|DH|}=3.$$

Linear Algebra I, Sheet 1, MT2021

Systems of linear equations; matrices and their algebra.

Main course

1. Use any method you can think of to decide which (if any) of the following systems of linear equations with real coefficients have no solutions, which have a unique solution (in which case, what is it?), which have infinitely many solutions.

$$\text{(i)}\left\{\begin{array}{rcl} 2x+4y-3z &=& 0 \\ x-4y+3z &=& 0 \\ 3x-5y+2z &=& 1 \end{array}\right. ; \quad \text{(ii)}\left\{\begin{array}{rcl} x+2y+3z &=& 0 \\ 2x+3y+4z &=& 1 \\ 3x+4y+5z &=& 2 \end{array}\right. ; \quad \text{(iii)}\left\{\begin{array}{rcl} x+2y+3z &=& 0 \\ 2x+3y+4z &=& 2 \\ 3x+4y+5z &=& 2 \end{array}\right. .$$

2. Let $A := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B := \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $C := \begin{pmatrix} -1 & 5 \\ 4 & 4 \end{pmatrix}$, $D := \begin{pmatrix} 1 & 4 & -3 \\ 2 & 4 & -2 \end{pmatrix}$, $E := \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $F := \begin{pmatrix} -1 & 5 & -6 \\ 3 & 4 & -1 \end{pmatrix}$. For which pairs $X, Y \in \{A, B, C, D, E, F\}$ is $X - 2Y$ defined? And when it is defined, calculate it.

3. Calculate the following matrix products:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}; \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}^2 .$$

4. Let A be the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(i) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if A is diagonal (that is, $b = c = 0$).

(ii) Which 2×2 matrices A commute with $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$?

(iii) Use the results of (i) and (ii) to find the matrices A that commute with every 2×2 matrix.

5. For each $\alpha \in \mathbb{R}$, find the RRE form for the matrix

$$\begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 2 \\ 1 & 4 & -7 & 4 & 0 & \alpha \end{pmatrix}.$$

Use your result either to solve the following system of linear equations over $\mathbb{R}$, or to find values of α for which it has no solution:

$$\left\{\begin{array}{rcrcrcrcrcl} x_1 &+& 2x_2 &-& 3x_3 &-& 2x_4 &+& 4x_5 &=& 1 \\ 2x_1 &+& 5x_2 &-& 8x_3 &-& x_4 &+& 6x_5 &=& 2 \\ x_1 &+& 4x_2 &-& 7x_3 &+& 4x_4 && &=& \alpha \end{array}\right.$$

6. Let A and B denote square matrices with real entries. For each of the following assertions, find either a proof or a counterexample.
(i) $A^2 - B^2 = (A - B)(A + B)$.
(ii) If $AB = 0$ then $A = 0$ or $B = 0$.
(iii) If $AB = 0$ then A and B cannot both be invertible.
(iv) If A and B are invertible then $A + B$ is invertible.
(v) If $ABA = 0$ and B is invertible then $A^2 = 0$.
[Hint: where the assertions are false there are counterexamples of size 2×2.]

Starter

S1. Let $A := \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 7 & -2 \end{pmatrix}$, $B := \begin{pmatrix} 4 & -3 & 1 \\ 3 & -2 & 1 \\ 7 & 0 & 1 \end{pmatrix}$, $C := \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 5 & -4 & 3 \end{pmatrix}$. For which pairs X, $Y \in \{A, B, C\}$ is XY defined? When it is defined, calculate it.

S2. Calculate the following matrix products.

$$\begin{pmatrix} x & y & z & w \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}; \quad \begin{pmatrix} x & y & z & w \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

S3. Let A, B be invertible $n \times n$ matrices. Then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Pudding

P1. Let $\alpha \in \mathbb{R}$ and

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

(i) Show that $A^2 = I_2$.
(ii) Show that there is no real 2×2 matrix M such that $M^2 = A$.
(iii) Is there a matrix M with complex entries such that $M^2 = A$?

P2. We can identify a complex number $z = x + yi$ with the 2×2 real matrix Z by

$$x + yi = z \quad \leftrightarrow \quad Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Let z, w be two complex numbers such that $z \leftrightarrow Z$ and $w \leftrightarrow W$. Show that $z + w \leftrightarrow Z + W$ and $zw \leftrightarrow ZW$.

P3. (Here $\mathbb{Z}_n$ denotes the integers modulo n.) Put the following matrix into reduced row echelon form when working over (i) $\mathbb{R}$; (ii) $\mathbb{Z}_3$; (iii) $\mathbb{Z}_5$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix}$$

How many solutions (x_1, x_2, x_3, x_4) are there to the linear system

$$x_1 + x_2 + x_3 = 1; \qquad x_1 + 2x_2 - x_3 + 2x_4 = 0; \qquad x_2 + x_3 + 2x_4 = 1;$$

when working over (i) $\mathbb{R}$; (ii) $\mathbb{Z}_3$ (iii) $\mathbb{Z}_5$?

Linear Algebra I, Sheet 2, MT2021

More about matrices. Elementary row operations; echelon form of a matrix.

Main course

1. Let J_n be the $n \times n$ matrix with all entries equal to 1. For what values of $\alpha,\, \beta \in \mathbb{R}$ is the matrix $\alpha I_n + \beta J_n$ is invertible?
[Hint: note that $J_n^2 = nJ_n$; seek an inverse of $\alpha I_n + \beta J_n$ of the form $\lambda I_n + \mu J_n$ where $\lambda,\, \mu \in \mathbb{R}$.]

Find the inverse of $\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$.

2. Use EROs to reduce each of the following matrices to RRE form:

$$\text{(a)} \begin{pmatrix} 2 & 4 & -3 & 0 \\ 1 & -4 & 3 & 0 \\ 3 & -5 & 2 & 1 \end{pmatrix}; \quad \text{(b)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{pmatrix}; \quad \text{(c)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix}.$$

3. Let A and B be $m \times n$ matrices, and let C be an $n \times p$ matrix.
(i) Show that $(A+B)^T = A^T + B^T$ and that $(\lambda A)^T = \lambda A^T$ for scalars λ.
(ii) Show that $(AC)^T = C^T A^T$.
(iii) Suppose that $m = n$ and A^{-1} exists. Show that A^T is invertible and that $(A^T)^{-1} = (A^{-1})^T$.

4. The *trace* of a square matrix is the sum of its diagonal elements. Let A be an $m \times n$ matrix and B be $n \times m$. Show that

$$\operatorname{trace}(AB) = \operatorname{trace}(BA).$$

5. Use EROs to find the inverses of each of the following matrices

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

6. (i) Show that if the $m \times n$ matrices A, B can be reduced to the same matrix E in echelon form, then there is a sequence of EROs that changes A into B.
(ii) Show that an $n \times n$ real matrix may be reduced to RRE form by a sequence of at most n^2 EROs.

Starter

S1. Use EROs to determine whether the following matrices are invertible. For each invertible matrix, find the inverse.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix}; \qquad \begin{pmatrix} -2 & 0 & 4 & 3 \\ 1 & 7 & 5 & -6 \\ -3 & 7 & 13 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

S2. Show that the inverse of an ERO is an ERO. (Hint: consider each of the three types of ERO separately.)

S3. Prove from the vector space axioms that if V is a vector space, v, $z \in V$ and $v + z = v$, then $z = 0_V$.

Pudding

P1. Use EROs to explore for which real numbers a, b, c, d the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. You will need to proceed on a case by case basis.

P2.

(i) What happens if we multiply an upper triangular matrix by an upper triangular matrix?
(ii) What if we multiply a lower triangular matrix by a lower triangular matrix?
(iii) What if we multiply a lower triangular matrix by an upper triangular matrix?

P3. Let $V := \mathbb{R} \times \mathbb{Z}$, the set of all pairs (x, k) where x is a real number and k is an integer. Define addition coordinatewise so that $(x, k) + (y, m) = (x + y, k + m)$, and define scalar multiplication by real numbers λ by the rule $\lambda(x, k) = (\lambda x, 0)$. Which of the vector space axioms are satisfied, and which are not?

STEP Support Programme

STEP 2 Vectors Topic Notes

Notation

Vectors can be written as column vectors, or by using **i**, **j**, **k** notation. We have:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Magnitude

The magnitude of a vector is the length of the vector. If $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{x}$ then:

$$|\mathbf{x}| = \sqrt{a^2 + b^2 + c^2}$$

Scalar product

The scalar (or dot) product of two vectors is given by:

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = ax + by + cz$$

We can express the magnitude using $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$.

The scalar product can also be expressed as:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between the two vectors.

Position vectors

Usually $\mathbf{a}$ is the position vector of point A with respect to a fixed point, usually the origin, O. We can write $\mathbf{a} = \overrightarrow{OA}$. To find the vector $\overrightarrow{AB}$ we can think about going backwards along OA and then along OB. This gives:

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

Distance between two points

The distance between points A and B can be found by using:

$$|\overrightarrow{AB}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$$

Vector equations of lines

The vector equation of a line which passes through a point A with position vector $\mathbf{a}$ and travels in the direction described by vector $\mathbf{d}$ is:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

In particular, the vector equation of a line passing through A and B can be written as $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, or by rearranging (and as is often used in STEP)

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$$

If $0 < \lambda < 1$ then the point on the line is between A and B, and if λ is outside this range then the point is on the line through A and B but not between them.

Two lines in **2 dimensions** either meet at a point, or are parallel, or are the same line.

In **3 dimensions** the two lines can meet at a point, be the same line, be parallel or they can be *skew* which is to say that they are not parallel, but also do not meet.

To show that two lines are parallel you need to show that the direction vectors are multiples of each other. If the direction vectors are multiples of each other and you can find a point in common then the two lines are the same line.

Finding the intersection of two lines can be done by equating components. Consider the lines $\mathbf{r_1} = \mathbf{a} + \lambda\mathbf{d}$ and $\mathbf{r_2} = \mathbf{b} + \mu\mathbf{m}$.

In the case where they are not parallel, where they meet we have:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

This gives three equations (one for the x component etc.). You can use two of these to find λ and μ, and then need to check to see if these satisfy the third equation. If they do satisfy all three equations then the lines meet at one point, otherwise the lines are skew.

Top Tips!

- Dot things together, and use the fact that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$
- $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
- $(\lambda\mathbf{a} + \mu\mathbf{b}) \cdot \mathbf{c} = \lambda(\mathbf{a} \cdot \mathbf{c}) + \mu(\mathbf{b} \cdot \mathbf{c})$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \geq 0$
- $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular or zero
- If $\mathbf{a}$ is a unit vector then $|\mathbf{a}| = 1$

STEP Support Programme

STEP 2 Vectors Questions

1 **2002 S2 Q7**

In 3-dimensional space, the lines m_1 and m_2 pass through the origin and have directions $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$, respectively. Find the directions of the two lines m_3 and m_4 that pass through the origin and make angles of $\pi/4$ with both m_1 and m_2. Find also the cosine of the acute angle between m_3 and m_4.

The points A and B lie on m_1 and m_2 respectively, and are each at distance $\lambda\sqrt{2}$ units from O. The points P and Q lie on m_3 and m_4 respectively, and are each at distance 1 unit from O. If all the coordinates (with respect to axes $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) of A, B, P and Q are non-negative, prove that:

(i) there are only two values of λ for which AQ is perpendicular to BP;

(ii) there are no non-zero values of λ for which AQ and BP intersect.

2 **2011 S2 Q5**

The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O, and O, A and B are non-collinear. The point C, with position vector $\mathbf{c}$, is the reflection of B in the line through O and A. Show that $\mathbf{c}$ can be written in the form

$$\mathbf{c} = \lambda\mathbf{a} - \mathbf{b}$$

where $\lambda = \dfrac{2\,\mathbf{a.b}}{\mathbf{a.a}}$.

The point D, with position vector $\mathbf{d}$, is the reflection of C in the line through O and B. Show that $\mathbf{d}$ can be written in the form

$$\mathbf{d} = \mu\mathbf{b} - \lambda\mathbf{a}$$

for some scalar μ to be determined.

Given that A, B and D are collinear, find the relationship between λ and μ. In the case $\lambda = -\frac{1}{2}$, determine the cosine of $\angle AOB$ and describe the relative positions of A, B and D.

3 **2009 S2 Q8**

The non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, given by

$$\mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b} \quad \text{and} \quad \mathbf{q} = \mu\mathbf{a} + (1-\mu)\mathbf{c}$$

where $0 < \lambda < 1$ and $\mu > 1$. Draw a diagram showing A, B, C, P and Q.

Given that $CQ \times BP = AB \times AC$, find μ in terms of λ, and show that, for all values of λ, the the line PQ passes through the fixed point D, with position vector $\mathbf{d}$ given by $\mathbf{d} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$. What can be said about the quadrilateral $ABDC$?

4 **2010 S2 Q5**

The points A and B have position vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $5\mathbf{i} - \mathbf{j} - \mathbf{k}$, respectively, relative to the origin O. Find $\cos 2\alpha$, where 2α is the angle $\angle AOB$.

(i) The line L_1 has equation $\mathbf{r} = \lambda(m\mathbf{i} + n\mathbf{j} + p\mathbf{k})$. Given that L_1 is inclined equally to OA and to OB, determine a relationship between m, n and p. Find also values of m, n and p for which L_1 is the angle bisector of $\angle AOB$.

(ii) The line L_2 has equation $\mathbf{r} = \mu(u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$. Given that L_2 is inclined at an angle α to OA, where $2\alpha = \angle AOB$, determine a relationship between u, v and w.

Hence describe the surface with Cartesian equation $x^2 + y^2 + z^2 = 2(yz + zx + xy)$.

STEP Support Programme

STEP 3 Vectors Topic Notes

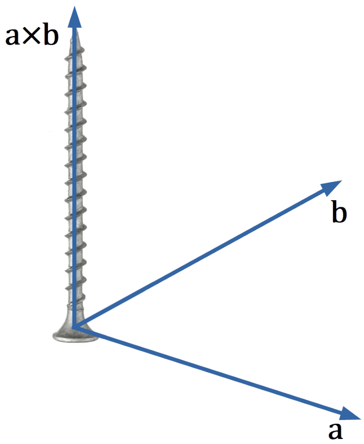
Vector Product

The vector product of the two vectors $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and θ is the angle between them.

The direction of $\hat{\mathbf{n}}$ is given by the *right hand rule* — if $\mathbf{a}$ is the thumb and $\mathbf{b}$ is the index finger then the direction of $\hat{\mathbf{n}}$ is given by the direction of the second finger[1]. Alternatively, you can consider a normal (right-handed) screw. If you turn the screw from $\mathbf{a}$ to $\mathbf{b}$ then the direction that the screw moves in is the direction of $\mathbf{a} \times \mathbf{b}$.



If you move your thumb to where the index finger was and the index finger to where the thumb was you should find that your second finger is now pointing in the opposite direction. With the screw analogy, if you turn in the opposite direction the screw will loosen and move downwards. This means that:

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Note that $\mathbf{a} \times \mathbf{a} = 0$ as in this case we have $\theta = 0$.

If $\mathbf{a} \times \mathbf{b} = 0$ then either $\mathbf{a} = 0$, or $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are parallel (i.e. one is a multiple of the other).

[1] This assumes that you do not have hyper-mobile joints or any broken fingers.

In determinant form we have:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{or} \quad \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix}$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

The *Scalar Triple Product* is given by:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and is unchanged by a circular shift of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. We also have $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

Vector equation of a line

Let $\mathbf{r}$ be a general point on a line that passes through point A with position vector $\mathbf{a}$ and which is parallel to the vector $\mathbf{b}$. Then we can write the equation of the line as:

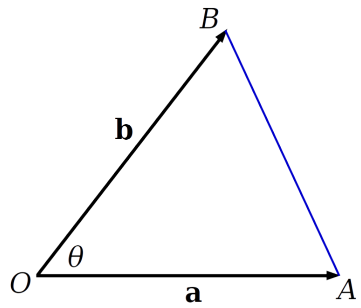
$$(\mathbf{r}-\mathbf{a})\times\mathbf{b} = 0 \quad \text{or} \quad \mathbf{r}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}\,.$$

This is because the vector $\mathbf{r}-\mathbf{a}$ is parallel to $\mathbf{b}$, and the vector product of two parallel vectors is equal to 0.

Areas and Volumes

Area of a triangle

Consider a triangle with vertices O, A and B.



Let the position vector of point A be $\mathbf{a}$ and the position vector of point B be $\mathbf{b}$. Also let the angle between $\mathbf{a}$ and $\mathbf{b}$ be θ. The area of the triangle is given by:

$$\tfrac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

(Remembering that $\hat{\mathbf{n}}$ is a unit vector, so $|\hat{\mathbf{n}}| = 1$).

If the triangle had vertices A, B and C, then the vectors of the lengths relative to C are $\overrightarrow{CA} = \mathbf{a}-\mathbf{c}$ and $\overrightarrow{CB} = \mathbf{b}-\mathbf{c}$. If the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is θ then the area of the triangle is:

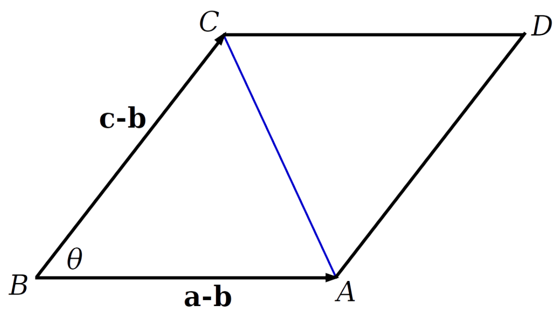
$$\tfrac{1}{2}|\overrightarrow{CA}||\overrightarrow{CB}|\sin\theta = \tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})|$$

Rearranging this gives:

$$\begin{aligned}\tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})| &= \tfrac{1}{2}|(\mathbf{a}\times\mathbf{b})-(\mathbf{c}\times\mathbf{b})-(\mathbf{a}\times\mathbf{c})+(\mathbf{c}\times\mathbf{c})| \\ &= \frac{1}{2}|(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})|\end{aligned}$$

Which is a nicely symmetrical result. You will not be expected to be able to quote this — however deriving this from $\tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})|$ would be fair game!

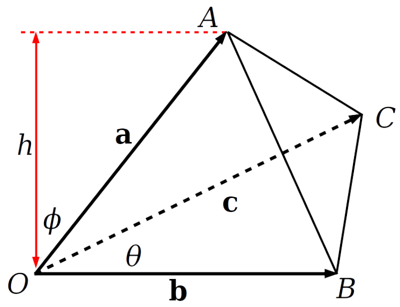
Area of a parallelogram Consider a parallelogram with vertices A, B, C and D.



The area of the parallelogram is twice the area of triangle ABC and so is equal to $||(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{b})|$.

Volume of a triangular based pyramid

Consider a pyramid with vertices O, A, B and C.



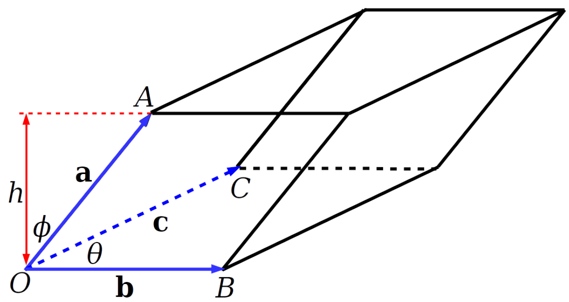
Let θ be the angle between OB and OC, and let ϕ be the angle between OA and the perpendicular height.

The volume of the pyramid is given by $\frac{1}{3} \times$ (area of the base) $\times\, h$. The area of the base is given by $\frac{1}{2}|\mathbf{b} \times \mathbf{c}|$. The height of the pyramid is given by $|\mathbf{a}|\cos\phi$. Using $\mathbf{p}\cdot\mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\phi$ we have:

$$\begin{aligned}\text{Volume } &= \tfrac{1}{3} \times |\mathbf{a}|\cos\phi \times \tfrac{1}{2}|\mathbf{b}\times\mathbf{c}| \\ &= \tfrac{1}{6}\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\end{aligned}$$

Volume of a parallelepiped

The volume of a parallelepiped (shown below) is $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. This can be derived in a similar way to the formula for the triangular based pyramid above.



Equation of a plane

- Vector $\mathbf{n}$ is perpendicular to plane Π, and plane Π passes through the point A with position vector $\mathbf{a}$. The equation of the plane can be written as $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n} = 0$ or equivalently $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$.
- If we write $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ can be written as $ax + by + cz = d$ where $d = \mathbf{a}\cdot\mathbf{n}$.
- The equation of a plane Π can also be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where $\mathbf{a}$ is the position vector of a point A in the plane, and $\mathbf{b}$ and $\mathbf{c}$ are two non-parallel vectors in the plane.

Other stuff on planes

- If two planes have normals $\mathbf{n}_1$ and $\mathbf{n}_2$ then the angle between the planes is given by:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

- The angle between a line with equation $\mathbf{r}_1 = \mathbf{a} + \lambda\mathbf{b}$ and the plane with equation $\mathbf{r}_2 \cdot \mathbf{n} = p$ is given by:

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$
$$\implies \sin\theta = \frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$

  Note that the scalar product will involve the angle between the *normal* to the plane and the direction of the line. If the angle between the plane and the line is θ, then the angle between the normal to the plane and the line is $\frac{\pi}{2} - \theta$.

Intersection of a line and a plane

There are three possible situations here:

1 : There is a unique solution, so the line intersects the plane at a single point
2 : There are no solutions, i.e. the line never meets the plane — the line is parallel to the plane
3 : There are infinitely many solutions — in this case the line is contained in the plane

- **Example 1**

  Find the intersection(s) of the plane $\Pi : 2x + y - 2z = 4$ and the line $x - 1 = \dfrac{y-4}{2} = \dfrac{z+2}{3}$.

  Start by writing $x - 1 = \dfrac{y-4}{2} = \dfrac{z+2}{3} = \lambda$. We then have $x = \lambda + 1$, $y = 2\lambda + 4$ and $z = 3\lambda - 2$. Substituting these into the equation for the plane gives:

$$2(\lambda+1) + (2\lambda+4) - 2(3\lambda-2) = 4 \implies \lambda = 3$$

  And so the point of intersection is $(4, 10, 7)$.

  It is a very good idea to check these values in the original equations for the line and the plane — this point should satisfy both equations.

  If you try to find the points of intersection of the line equations $x - 1 = \dfrac{y-4}{2} = \dfrac{z+2}{2}$ and $x - 1 = \dfrac{y+2}{2} = \dfrac{z+2}{2}$ with the plane $\Pi : 2x + y - 2z = 4$ you should end up with $10 = 4$ in the first case (no solutions) and $4 = 4$ in the second (infinitely many solutions).

- **Example 2**
  Find any intersection(s) of the line $\mathbf{r} = \mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and the plane $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 10$.

  Substituting in the equation of the line gives:

$$\begin{aligned} \begin{pmatrix} 1+\lambda \\ -\lambda \\ 1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} &= 10 \\ 3 + 3\lambda - \lambda - 1 - 2\lambda &= 10 \\ 2 &= 10 \end{aligned}$$

  So there are no points of intersection — the line is parallel to the plane.

Intersection of two planes

Here you can write both planes in Cartesian form, and then eliminate one of the variables.

Example:
Find the equations of the line of intersection of the planes $\Pi_1 : \mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = 5$ and $\Pi_2 : \mathbf{r} \cdot (\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 7$.

We can write the equations of the planes as $2x + y - z = 5$ and $x - 3y + z = 7$. Eliminating z gives $3x - 2y = 12$ i.e. $y = \frac{3}{2}x - 6$. Substituting this into Π_1 gives $z = 2x + y - 5$ i.e. $z = \frac{7}{2}x - 11$.

Setting $x = \lambda$ gives the equation of the line as $\mathbf{r} = -6\mathbf{j} - 11\mathbf{k} + \lambda(\mathbf{i} + \frac{3}{2}\mathbf{j} + \frac{7}{2}\mathbf{k})$.

If the normals of the two plane equations are multiples of each other then the two planes are parallel and either never intersect, or are actually the same plane.

Perpendicular distances

- **Distance between two Parallel lines**
  If the two lines are parallel then they can be written as $\mathbf{r_1} = \mathbf{a_1} + \lambda\mathbf{b}$ and $\mathbf{r_2} = \mathbf{a_2} + \mu\mathbf{b}$. Therefore the vector between a general point on $\mathbf{r_1}$ and $\mathbf{r_2}$ can be written as $(\mathbf{a_1} - \mathbf{a_2}) + t\mathbf{b}$, and then this can then be minimised over t to find the shortest (perpendicular) distance.

  Example:
  Find the distance between the lines $\mathbf{r_1} = 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{r_2} = \mathbf{i} + \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

  We can write the vector of the line segment connecting two points on the lines as:

$$\begin{aligned} \begin{pmatrix} 2\lambda \\ 2-\lambda \\ 3+\lambda \end{pmatrix} - \begin{pmatrix} 1+2\mu \\ -\mu \\ 1+\mu \end{pmatrix} &= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + (\lambda - \mu) \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

So we want to minimise $\sqrt{(2t-1)^2+(2-t)^2+(t+2)^2} = \sqrt{6t^2-4t+9}$ which is minimised when $t=\frac{1}{3}$ (by differentiation of the quadratic). This gives the minimum/perpendicular distance as:

$$\sqrt{\frac{6}{9}-\frac{4}{3}+9} = \sqrt{\frac{25}{3}} = \frac{5\sqrt{3}}{3}$$

- **Distance between a point and a line**
  Use the same method as above!

  Example
  Find the distance between the point $\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and the line with equation $\mathbf{r} = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{k})$.

  The vector of the line segment joining the point and a general point on the line is given by:

$$\begin{pmatrix}2\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1+2\lambda\\-3\\3-\lambda\end{pmatrix}$$

  So we want to minimise $\sqrt{(1+2\lambda)^2+(-3)^2+(3-\lambda)^2} = \sqrt{5\lambda^2-2\lambda+9}$. This is minimised when $\lambda=\frac{1}{5}$.

- **Distance between a point and a plane**
  If you know the normal to the plane, then the shortest distance to a point on the plane will be along the same direction as the normal.

  Example:
  Find the distance of the point $(1,3,-2)$ to the plane $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = 6$.

  The perpendicular line joining the point to the plane has equation:

$$\begin{pmatrix}1\\3\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$$

  and where this meets the plane we have:

$$\begin{aligned}\begin{pmatrix}1+\lambda\\3+2\lambda\\-2+2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} &= 6\\ (1+\lambda)+2(3+2\lambda)+2(-2+2\lambda) &= 6\\ 9\lambda+3 &= 6\\ \lambda &= \frac{1}{3}\end{aligned}$$

  So the line meets the plane at the point:

$$\begin{pmatrix}1\\3\\-2\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

  and this distance of this point from the point $(1,3,-2)$ is:

$$\frac{1}{3}\left|\begin{pmatrix}1\\2\\2\end{pmatrix}\right| = \frac{1}{3}\times 3 = 1$$

- **Distance between two Skew lines**

  If two lines are *skew* (that is they are not parallel and do not meet) then the shortest distance between them will be given by the line segment XY where X lies on one line, Y lies on the other and XY is perpendicular to both lines. If the two lines are $\mathbf{r_1} = \mathbf{a_1} + \lambda\mathbf{b_1}$ and $\mathbf{r_2} = \mathbf{a_2} + \lambda\mathbf{b_2}$ then a vector perpendicular to them both will be given by $\mathbf{b_1} \times \mathbf{b_2}$. You can then find a vector of the line segment between a general point on each line and show where this is parallel to $\mathbf{b_1} \times \mathbf{b_2}$.

  Example:

  Find the distance between the lines $\mathbf{r_1} = \mathbf{i}+\mathbf{j}+\lambda(2\mathbf{i}-\mathbf{j}+5\mathbf{k})$ and $\mathbf{r_2} = -\mathbf{i}+\mathbf{j}+2\mathbf{k}+\mu(2\mathbf{i}-5\mathbf{j}+\mathbf{k})$.

  A vector perpendicular to both lines is given by:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 5 \\ 2 & -5 & 1 \end{vmatrix} = \begin{pmatrix} 24 \\ 8 \\ -8 \end{pmatrix}$$

  So we want to find points X and Y on the two lines so that the line XY is parallel to $24\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$ (or equivalently we could use $3\mathbf{i} + \mathbf{j} - \mathbf{k}$). Using the line equations we have:

$$\begin{pmatrix} 1+2\lambda \\ 1-\lambda \\ 5\lambda \end{pmatrix} - \begin{pmatrix} -1+2\mu \\ 1-5\mu \\ 2+\mu \end{pmatrix} = k\begin{pmatrix} 24 \\ 8 \\ -8 \end{pmatrix}$$

  This gives us three simultaneous equations:

$$\begin{aligned} 2+2\lambda-2\mu &= 24k && (1) \\ 5\mu-\lambda &= 8k && (2) \\ 5\lambda-\mu-2 &= -8k && (3) \end{aligned}$$

  Then $(1)+2(2)$ gives $2+8\mu = 40k \implies 4\mu+1 = 20k$ and $(3)+5(2)$ gives $24\mu-2 = 32k \implies 12\mu-1 = 16k$. Substituting for 4μ gives $3(20k-1)-1 = 16k \implies 44k = 4 \implies k = \frac{1}{11}$.

  The perpendicular distance between the two lines is therefore:

$$\left|\frac{1}{11}\begin{pmatrix} 24 \\ 8 \\ -8 \end{pmatrix}\right| = \frac{8}{11}\sqrt{3^2+1^2+1^2} = \frac{8\sqrt{11}}{11}$$

  There is a formula for the perpendicular distance between two skew lines — $\left|\dfrac{(\mathbf{a_1}-\mathbf{a_2})\cdot(\mathbf{b_1}\times\mathbf{b_2})}{|\mathbf{b_1}\times\mathbf{b_2}|}\right|$ — however I prefer to use the above method as it is easier to remember! In both this case and the method shown above if the lines actually meet then the methods will give a distance of 0 (and the first method can then be used to find the point of intersection).

STEP Support Programme

STEP 3 Vectors Questions

1 SPECIMEN S2 Q9

(i) Let $\mathbf{a}$ and $\mathbf{b}$ be given vectors with $\mathbf{b} \neq \mathbf{0}$, and let $\mathbf{x}$ be a position vector. Find the condition for the sphere $|\mathbf{x}| = R$, where $R > 0$, and the plane $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ to intersect.

When this condition is satisfied, find the radius and the position vector of the centre of the circle in which the plane and sphere intersect.

(ii) Let $\mathbf{c}$ be a given vector, with $\mathbf{c} \neq \mathbf{0}$. The vector $\mathbf{x}'$ is related to the vector $\mathbf{x}$ by

$$\mathbf{x}' = \mathbf{x} - \frac{2(\mathbf{x} \cdot \mathbf{c})\mathbf{c}}{|\mathbf{c}|^2}.$$

Interpret this relation geometrically.

2 92 S2 Q9

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C in three-dimensional space. Suppose that A, B, C and the origin O are not all in the same plane. Describe the locus of the point whose position vector $\mathbf{r}$ is given by

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where λ and μ are scalar parameters. By writing this equation in the form $\mathbf{r} \cdot \mathbf{n} = p$ for a suitable vector $\mathbf{n}$ and scalar p, show that

$$-(\lambda + \mu)\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \lambda\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) + \mu\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

for all scalars λ, μ.

Deduce that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

Say briefly what happens if A, B, C and O are all in the same plane.

3 **93 S2 Q4**

Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\mathbf{m}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\mathbf{m}_2$ respectively, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{m}_1 \times \mathbf{m}_2)|}{|(\mathbf{m}_1 \times \mathbf{m}_2)|}.$$

(i) Find the shortest distance between the lines in the case

$$\mathbf{p}_1 = (2, 1, -1) \qquad \mathbf{p}_2 = (1, 0, -2) \qquad \mathbf{m}_1 = \tfrac{1}{5}(4, 3, 0) \qquad \mathbf{m}_2 = \tfrac{1}{\sqrt{10}}(0, -3, 1).$$

(ii) Two aircraft, A_1 and A_2, are flying in the directions given by the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ at constant speeds v_1 and v_2. At time $t = 0$ they pass the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2\,|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - [(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1. How close is that, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}_1$ and $\mathbf{m}_2$ are as in **(i)**?

4 **95 S3 Q8**

A plane π in 3-dimensional space is given by the vector equation $\mathbf{r} \cdot \mathbf{n} = p$, where $\mathbf{n}$ is a unit vector and p is a non-negative real number. If $\mathbf{x}$ is the position vector of a general point X, find the equation of the normal to π through X and the perpendicular distance of X from π.

The unit circles C_i, $i = 1, 2$, with centres $\mathbf{r}_i$, lie in the planes π_i given by $\mathbf{r} \cdot \mathbf{n}_i = p_i$, where the $\mathbf{n}_i$ are unit vectors, and p_i are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number λ such that

$$\mathbf{r}_1 + \lambda\mathbf{n}_1 = \mathbf{r}_2 \pm \lambda\mathbf{n}_2.$$

Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0$$

and that

$$(p_1 - \mathbf{n}_1 \cdot \mathbf{r}_2)^2 = (p_2 - \mathbf{n}_2 \cdot \mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.

STEP Support Programme

STEP 2 Matrices Questions

This collection of questions is different from most of the STEP Support Programme, since matrices have not been on the STEP syllabus for many years. Note the following:

- These are **not** past STEP questions; they are from old A-level papers and similar.
- Many of these questions are *longer* or *shorter* than a typical STEP question.
- Many of these questions are *easier* or *harder* than a typical STEP question.
- The questions appear in chronological order of their origin, **not** in approximate order of difficulty.

The questions have been chosen to challenge you to think about matrices in a more sophisticated way than current A-level questions are likely to, and so will be a good preparation for what might appear on future STEP papers.

Matrices will first be examinable on STEP papers 2 and 3 from 2019 (under the new specification). There were a small number of STEP questions on the topic of matrices in the 1980s and 1990s. These can be found by searching for 'matrices' on the STEP database. Some of these are appropriate for today's STEP paper 2 or 3, but others require content beyond the current specification.

Acknowledgements (copyright details and question sources) for the questions in this module appear on the final page.

1 The complex number $x + \mathrm{i}y$ is mapped into the complex number $X + \mathrm{i}Y$ where X and Y are given by the equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Which numbers are invariant under the mapping?

2 The simultaneous equations

$$\begin{aligned} x + 2y &= 4, \\ 2x - y &= 0, \\ 3x + y &= 5 \end{aligned}$$

may be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}, \quad \text{or} \quad \mathbf{AX} = \mathbf{B}.$$

Carry out numerically the procedure of the three following steps:

(1) $\mathbf{A}^{\mathrm{T}}\mathbf{AX} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$;

(2) $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{AX} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B}$;

(3) $\mathbf{IX} = \begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B}$.

Verify that the values of x, y so found do not satisfy all the original three equations. Suggest a reason for this.

Under what circumstances will the procedure given above, when applied to a set of three simultaneous equations in two variables, result in values which satisfy the equations?

Note: The final part of this question is challenging to answer fully; a complete solution is beyond what would be expected on a STEP examination.

3 Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be real 2×2 matrices and write

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}, \text{ etc.}$$

Prove that:

(i) $[\mathbf{A}, \mathbf{A}] = \mathbf{O}$, where $\mathbf{O}$ is the zero matrix,

(ii) $[[\mathbf{A}, \mathbf{B}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] = \mathbf{O}$,

(iii) if $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$, then $[\mathbf{A}, \mathbf{B}^m] = m\mathbf{B}^{m-1}$ for all positive integers m.

At each step you should state clearly any properties of matrices which you use.

The *trace*, $\mathrm{Tr}(\mathbf{A})$, of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is defined by

$$\mathrm{Tr}(\mathbf{A}) = a_{11} + a_{22}.$$

Prove that:

(iv) $\mathrm{Tr}(\mathbf{A} + \mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B})$,

(v) $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$,

(vi) $\mathrm{Tr}(\mathbf{I}) = 2$.

Deduce that there are no matrices satisfying $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$. Does this in any way invalidate the statement in (iii)?

4 Matrices $\mathbf{P}$ and $\mathbf{Q}$ are given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$

(where $\mathrm{i}^2 = -1$). Show that $\mathbf{P}^2 = \mathbf{Q}^2$, $\mathbf{PQP} = \mathbf{Q}$ and $\mathbf{P}^4 = \mathbf{I}$, the identity matrix. Deduce that, for all positive integers n, $\mathbf{P}^n\mathbf{QP}^n = \mathbf{Q}$. Hence, or otherwise, show that if $\mathbf{X}$ and $\mathbf{Y}$ are each matrices of the form

$$\mathbf{P}^m\mathbf{Q}^n, \quad m = 1, 2, 3, 4; \; n = 1, 2$$

then $\mathbf{XY}$ has the same form.

5 (a) Show that if $\mathbf{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then

$$\mathbf{A}^2 - (p+s)\mathbf{A} + (ps - qr)\mathbf{I} = \mathbf{O},$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{O}$ is the zero matrix.

(b) Given that $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that $\mathbf{X}^2 = \mathbf{O}$, show that $\mathbf{X}$ can be written either in terms of a and b only or in terms of c only, or of b only.

Show that when $\mathbf{X}$ is written in terms of c only, the solution can be written in the form:

$$\mathbf{X} = c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and interpret this result in terms of transformations of the plane represented by these matrices, relating your answer to the fact that $\mathbf{X}^2 = \mathbf{O}$.

6 A mapping $(x, y) \to (u, v)$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Show briefly that this mapping is not one to one.

Find the locus, L, of all points which map to $(1, -4)$. Describe the locus of (u, v) as (x, y) is allowed to vary throughout the plane. Show that any given point, P, on this locus is the image of just one point on the y-axis, and describe how the set of all points with image P is related to the locus L.

7 You are given that $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are 2×2 matrices, $\mathbf{I}$ is the identity matrix and $\mathbf{P}^{-1}$ exists.

(i) Prove, by expanding both sides, that

$$\det(\mathbf{PQ}) = \det\mathbf{P}\det\mathbf{Q}.$$

Deduce that

$$\det(\mathbf{P}^{-1}\mathbf{Q} + \mathbf{I}) = \det(\mathbf{QP}^{-1} + \mathbf{I}).$$

(ii) If $\mathbf{PX} = \mathbf{XP}$ for *every* 2×2 matrix $\mathbf{X}$, prove that $\mathbf{P} = \lambda\mathbf{I}$, where λ is a constant.

(iii) If $\mathbf{RQ} = \mathbf{QR}$, prove that

$$\mathbf{RQ}^n = \mathbf{Q}^n\mathbf{R} \quad \text{and} \quad \mathbf{R}^n\mathbf{Q}^n = \mathbf{Q}^n\mathbf{R}^n$$

for any positive integer n.

8 The real 3×3 matrix $\mathbf{A}$ is such that $\mathbf{A}^2 = \mathbf{A}$.

(i) Prove that $(\mathbf{I} - \mathbf{A})^2 = \mathbf{I} - \mathbf{A}$.

(ii) Express $(\mathbf{I} - \mathbf{A})^3$ in the form $\mathbf{I} + k\mathbf{A}$, where k is a number to be determined.

(iii) Prove that, for all real constants λ and all positive integers n,

$$(\mathbf{I} + \lambda\mathbf{A})^n = \mathbf{I} + \big((\lambda + 1)^n - 1\big)\mathbf{A}.$$

Use this result to verify your answer to (ii).

Acknowledgements

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In the list of sources below, the following abbreviations are used:

O&C	Oxford and Cambridge Schools Examination Board
SMP	School Mathematics Project
MEI	Mathematics in Education and Industry
QP	Question paper
Q	Question

1 O&C, A level Mathematics (SMP), 1966, QP Mathematics II, Q A3

2 O&C, A level Mathematics (SMP), 1967, QP Mathematics II, Q B22

3 O&C, A level Mathematics (MEI), 1968, QP MEI 20, Pure Mathematics III (Special Paper), Q 3; editorial changes here: the definition of **O** is inserted, the implication symbol is written in words, and the reference to the matrix ring is removed

4 O&C, A level Mathematics (MEI), 1968, QP MEI 143*, Pure Mathematics I, Q 6

5 O&C, A level Mathematics (MEI), 1980, QP 9655/1, Pure Mathematics 1, Q 2; editorial changes here: use **O** rather than **0** for the zero matrix, and define the notation.

6 O&C, A level Mathematics (MEI), 1981, QP 9655/1, Pure Mathematics 1, Q 6(b)

7 O&C, A level Mathematics (MEI), 1986, QP 9657/0, Mathematics 0 (Special Paper), Q 2; editorial change here: **I** is called the identity matrix rather than the unit matrix

8 O&C, A level Mathematics (MEI), 1987, QP 9650/2, Mathematics 2, Q 16

STEP Support Programme

STEP 3 Matrices Questions

This collection of questions is different from most of the STEP Support Programme, since matrices have not been on the STEP syllabus for many years. Note the following:

- These are **not** past STEP questions; they are from old A-level papers and similar.
- Many of these questions are *longer* or *shorter* than a typical STEP question.
- Many of these questions are *easier* or *harder* than a typical STEP question.
- The questions appear in chronological order of their origin, **not** in approximate order of difficulty.

The questions have been chosen to challenge you to think about matrices in a more sophisticated way than current A-level questions are likely to, and so will be a good preparation for what might appear on future STEP papers.

Matrices will first be examinable on STEP papers 2 and 3 from 2019 (under the new specification). There were a small number of STEP questions on the topic of matrices in the 1980s and 1990s. These can be found by searching for 'matrices' on the STEP database. Some of these are appropriate for today's STEP paper 2 or 3, but others require content beyond the current specification.

Acknowledgements (copyright details and question sources) for the questions in this module appear on the final page.

1 Express the determinant

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

as the product of factors which are linear in a, b, c.

Hence, or otherwise, find $x : y : z : u$ if

$$\begin{aligned} x + 2\ y + 3\ z + 4\ u &= 0, \\ x + 2^2 y + 3^2 z + 4^2 u &= 0, \\ x + 2^3 y + 3^3 z + 4^3 u &= 0. \end{aligned}$$

2 Prove that $(a - b)$ and $(x - y)$ are factors of the determinant

$$\begin{vmatrix} (a+x)^2 & (a+y)^2 & (a+z)^2 \\ (b+x)^2 & (b+y)^2 & (b+z)^2 \\ (c+x)^2 & (c+y)^2 & (c+z)^2 \end{vmatrix}$$

and factorise the determinant completely.

3 Given that α, β, γ are the roots of the equation

$$\begin{vmatrix} a & b & x \\ x & c & a \\ c & x & b \end{vmatrix} = 0,$$

prove that

$$\alpha^3 + \beta^3 + \gamma^3 = -6abc.$$

4 Prove that if the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{aligned}\right\}$$

are simultaneously satisfied by values of x, y, z which are not all zero, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Hence, or otherwise, eliminate x, y, z from the equations

$$a = \frac{x}{y-z}, \quad b = \frac{y}{z-x}, \quad c = \frac{z}{x-y}.$$

[Your answer should **not** be left in determinant form.]

5 A matrix $\mathbf{M}$ is said to be transposed into the matrix $\mathbf{M}^{\mathrm{T}}$ if the first row of $\mathbf{M}$ becomes the first column of $\mathbf{M}^{\mathrm{T}}$, the second row of $\mathbf{M}$ becomes the second column of $\mathbf{M}^{\mathrm{T}}$, and so on. Write down the transposes of the matrices

$$\mathbf{M} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ a & 0 & 0 \end{pmatrix}.$$

Calculate the matrix products $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ and $\mathbf{TM}$; show also that $(\mathbf{TM})^{\mathrm{T}} = \mathbf{M}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$.

If the elements of $\mathbf{M}$ are the Cartesian coordinates of a point P, what information is provided by the element of $\mathbf{M}^{\mathrm{T}}\mathbf{M}$?

If the matrix $\mathbf{T}$ describes a transformation of the points P of three dimensional space, interpret geometrically the equation

$$(\mathbf{TM})^{\mathrm{T}}(\mathbf{TM}) = \mathbf{M}^{\mathrm{T}}\mathbf{M},$$

and find all appropriate values of a, b and c for which this equation holds for all points P.

6 The functions $t \to \mathbf{P}$, $t \to \mathbf{Q}$ map real numbers t onto matrices $\mathbf{P}$, $\mathbf{Q}$ of fixed dimension; that is, each element of each matrix is a real function of t, and we suppose all the functions to be differentiable. A scalar multiplier s is also a function of t. The derivative of a matrix is defined as a matrix whose elements are the derivatives of the elements of the original matrix. We write $\mathrm{d}\mathbf{P}/\mathrm{d}t$ as $\dot{\mathbf{P}}$, $\mathrm{d}s/\mathrm{d}t$ as $\dot{s}$, and so on.

(i) Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}(s\mathbf{P}) = \dot{s}\mathbf{P} + s\dot{\mathbf{P}}.$$

(ii) If the product $\mathbf{PQ}$ is defined, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{PQ}) = \dot{\mathbf{P}}\mathbf{Q} + \mathbf{P}\dot{\mathbf{Q}}.$$

(iii) Prove that the derivative of a constant matrix is the zero matrix.

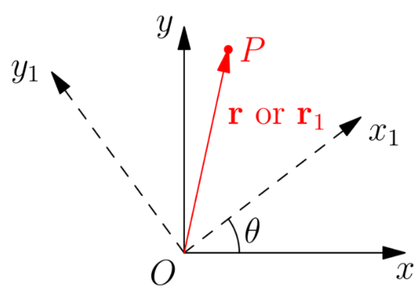
(iv) If $\mathbf{M}$ is the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

θ being a function of t, prove that $\dot{\mathbf{M}} = \mathbf{MJ}\dot{\theta}$, where

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The position of a particle in a plane is specified by a vector which can be described either as $\mathbf{r}$ relative to a coordinate system with rectangular axes Ox, Oy, or as $\mathbf{r}_1$ relative to a coordinate system with axes Ox_1, Oy_1, as shown in the diagram. The angle xOx_1, denoted by θ, is a function of t.



Write an equation connecting $\mathbf{r}$ with $\mathbf{r}_1$, and prove that

$$\dot{\mathbf{r}} = \mathbf{M}(\dot{\mathbf{r}}_1 + \mathbf{J}\mathbf{r}_1\dot{\theta}).$$

7 If

$$x = e^{kt}(a\cos\lambda t + b\sin\lambda t)$$

show that

$$\dot{x} = e^{kt}(a'\cos\lambda t + b'\sin\lambda t),$$

where

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} k & \lambda \\ -\lambda & k \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

and dot denotes differentiation with respect to t, and find an expression for $\ddot{x}$ with coefficients given in a similar way.

A particular integral solution to

$$\ddot{x} + 2p\dot{x} + q^2 x = e^{kt}(C\cos\lambda t + D\sin\lambda t)$$

is

$$x = e^{kt}(a\cos\lambda t + b\sin\lambda t).$$

Show that

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} (k^2-\lambda^2)+2pk+q^2 & 2k\lambda+2p\lambda \\ -2k\lambda-2p\lambda & (k^2-\lambda^2)+2pk+q^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

and hence write $\begin{pmatrix} a \\ b \end{pmatrix}$ in terms of $\begin{pmatrix} C \\ D \end{pmatrix}$.

Discuss any particular cases.

Note: The final part of this question is somewhat vague; a complete answer would not be expected on a STEP examination without a more explicit question.

8 Sketch the graph whose equation is

$$x^2 - y^2 = a^2 \quad (a > 0).$$

Prove that the point $P(a\cosh t, a\sinh t)$ lies on this graph for all real values of t; but that there are points of the graph which cannot be expressed in this form.

If U is the point of the graph for which $t = u$, find the area of the region $\mathscr{R}$ bounded by the curve, the line OU and the x-axis. (You may assume that $u > 0$.)

Prove that the transformation with matrix

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \qquad (\alpha \neq 0)$$

transforms the point P into another point of the curve. Into what region is $\mathscr{R}$ transformed by this? What is the area of the transformed region? Give reasons for your answers.

9 Show that the equations

$$\begin{aligned} 3x + 2y + z &= a - 1, \\ -2x + (a-2)y - az &= 2a, \\ 6x + ay + (a-2)z &= 3a - 6 \end{aligned}$$

have a solution, not necessarily unique, unless $a = \frac{2}{3}$.

Find the complete solution when $a = 0$ and when $a = 4$.

10 If z is the complex number $x + \mathrm{i}y$, $\mathrm{i} = \sqrt{-1}$, let $\mathbf{M}(z)$ denote the 2×2 matrix

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Prove that

$$\mathbf{M}(z + z') = \mathbf{M}(z) + \mathbf{M}(z')$$

and that

$$\mathbf{M}(zz') = \mathbf{M}(z)\mathbf{M}(z').$$

Hence, or otherwise, show that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}.$$

Hence find three real 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^3 = \mathbf{I}$ and a 2×2 matrix $\mathbf{B}$ such that $\mathbf{B}^2 + \mathbf{I} = \mathbf{B}$.

11 Using any method you wish, calculate the inverse $\mathbf{A}^{-1}$ of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 6 \end{pmatrix}.$$

Interpret geometrically in three dimensions the equations

$$\begin{aligned} x + y + 2z &= 4, \\ 2x + 3y + 3z &= 8, \\ 3x + 4y + \lambda z &= 7 + \lambda \end{aligned}$$

in the cases $\lambda = 6$ and $\lambda = 5$.

12 Using any method you prefer, and solving the problems in any order you prefer, find

(i) the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & 8 & 3 \end{pmatrix},$$

(ii) the solution of the simultaneous equations

$$\begin{aligned} x + 2y + 4z &= 3, \\ 3x + 5y + 7z &= 12, \\ 6x + 8y + 3z &= 31. \end{aligned}$$

The last of the three equations is replaced by

$$6x + 8y + az = b,$$

and it is found that the first two equations together with the new third one have more than one solution. Find a and b, and state a geometrical interpretation in three dimensions for these equations.

13 $\mathbf{A}$ is a 3×3 matrix whose elements are 0, 1 or -1 and each row and each column of $\mathbf{A}$ contains exactly one non-zero element. Prove that $\mathbf{A}^2$, $\mathbf{A}^3$, $\ldots$, $\mathbf{A}^n$ are all of the same form and deduce that $\mathbf{A}^h = \mathbf{I}$ for some positive integer $h \leqslant 48$.

Interpret the action of $\mathbf{A}$ on a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ geometrically.

Note: the final part of this question is somewhat vague and slightly beyond the requirements of the STEP 3 specification.

14 Find all possible solutions to

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ k \end{pmatrix} \qquad \text{(i)},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix},$$

stating explicitly the value of k that gives these solutions.

The equations

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

may each be regarded as the equations of three planes; give a geometrical interpretation of your solution of equation (i) in terms of these sets of planes.

Write down the equation of the line, L, of intersection of the planes

$$x + 2y - z = 2 \quad \text{and} \quad 2x + y + z = 1.$$

Find the equation of the plane through $(0, 1, 0)$ perpendicular to L and the coordinates of the points where this plane intersects the lines of intersection of

$$x + 2y - z = 2 \quad \text{and} \quad 5x + 4y + z = 3$$

and

$$2x + y + z = 1 \quad \text{and} \quad 5x + 4y + z = 3.$$

15 (i) Given $\mathbf{M} = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$, calculate $\mathbf{M}^2$ and $\mathbf{M}^3$.

Suggest a form for $\mathbf{M}^n$ and confirm your suggestion, using the method of proof by induction.

(ii) Prove that, for any $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{AB} = \mathbf{BA}$$

if and only if $(\mathbf{A} - k\mathbf{I})(\mathbf{B} - k\mathbf{I}) = (\mathbf{B} - k\mathbf{I})(\mathbf{A} - k\mathbf{I})$ for all values of the real number k.

(iii) Prove that, for any $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}^{\mathrm{T}}$ is the transpose of $\mathbf{A}$.

(iv) Prove that, if $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ symmetric matrices, then $\mathbf{AB}$ is symmetric if and only if $\mathbf{AB} = \mathbf{BA}$.

16 Let $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(i) The complex number $a + \mathrm{i}b$ is represented by the matrix $a\mathbf{I} + b\mathbf{J}$. Show that if w and z are two complex numbers represented by the matrices $\mathbf{P}$ and $\mathbf{Q}$ respectively, then $w + z$ is represented by $\mathbf{P} + \mathbf{Q}$ and wz is represented by $\mathbf{PQ}$.

(ii) The matrices $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$ respectively.

Express $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B}^{-1}$ and $\mathbf{AB}^{-1}$ in the form $a\mathbf{I} + b\mathbf{J}$.

Write down the complex numbers represented by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{AB}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$. Hence, or otherwise, show that

$$\arctan(\tfrac{1}{2}) + \arctan(\tfrac{1}{3}) = \tfrac{1}{4}\pi.$$

17 Show that a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $a_{11} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$, may be decomposed into the product of a *lower* triangular form matrix $\mathbf{L}$ and an *upper* triangular form matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$ where

$$\mathbf{L} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & l & 0 \\ a_{31} & m & n \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}.$$

A system of simultaneous linear equations, $\mathbf{Ax} = \mathbf{c}$, may be solved by writing the equations as $\mathbf{LUx} = \mathbf{c}$ and letting $\mathbf{Ux} = \mathbf{u}$. The vector $\mathbf{u}$ is determined from $\mathbf{Lu} = \mathbf{c}$ and the solution $\mathbf{x}$ is then found from $\mathbf{Ux} = \mathbf{u}$. Since both $\mathbf{L}$ and $\mathbf{U}$ are triangular, these two sets of equations can be solved directly by back substitution.[1]

Use this method to solve

$$\begin{aligned} x + y - z &= 2, \\ 3x + 2y + 5z &= 1, \\ 4x - y + 2z &= 0. \end{aligned}$$

[1] Back substitution: one of x, y and z appears on its own in an equation, so can be found immediately, then this can be substituted into another equation involving just this and one other unknown to find that second unknown, and these two can be substituted into the third equation to find the last unknown.

Acknowledgements

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O&C	Oxford and Cambridge Schools Examination Board
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QP	Question paper
Q	Question

1 UCLES, A level Mathematics, 1951, QP 190, Further Mathematics I, Q 2

2 UCLES, A level Mathematics, 1953, QP 188, Further Mathematics I, Q 2

3 UCLES, A level Mathematics, 1954, QP 188, Further Mathematics IV (Scholarship Paper), Q 1

4 UCLES, A level Mathematics, 1958, QP 437/1, Further Mathematics I, Q 1

5 O&C, A level Mathematics (SMP), 1968, QP SMP 34*, Mathematics II, Q 9; editorial change: clarify the last part of the question by adding the 'for all points P' part.

6 O&C, A level Mathematics (SMP), 1968, QP SMP 35, Mathematics III (Special Paper), Q 10; editorial changes here: the term 'order' is replaced by 'dimension'; the term 'frame of reference' has been replaced by 'coordinate system', and a diagram has been introduced to clarify the idea

7 O&C, A level Mathematics (MEI), 1969, QP MEI 32, Applied Mathematics III (Special Paper), Q 13; editorial changes: explained the dot notation, corrected a typographical error in the differential equation (missing dot) and made parentheses consistent

8 O&C, A level Mathematics (SMP), 1970, QP SMP 58, Further Mathematics IV, Q 6

9 UCLES, A level Mathematics, 1971, QP 842/0, Pure Mathematics (Special Paper), Q1

10 O&C, A level Mathematics (MEI), 1971, QP MEI 54, Pure Mathematics II, Q 4

11 UCLES, A level Mathematics, 1972, QP 848/0, Mathematics 0 (Special Paper), Q 12; editorial change: the matrix is called $\mathbf{A}$ rather than $\mathbf{a}$.

12 UCLES, A level Mathematics, 1972, QP 852/0, Further Mathematics 0 (Special Paper), Q 17

13 O&C, A level Mathematics (MEI), 1973, QP MEI 84, Pure Mathematics II, Q 4; editorial change: the row vector has have been replaced by a column vector

14 O&C, A level Mathematics (MEI), 1976, QP 128, Pure Mathematics I, Q 3

15 O&C, A level Mathematics (MEI), 1982, QP 9655/1, Pure Mathematics 1, Q 8

16 O&C, A level Mathematics (MEI), 1985, QP 9658/1, Further Mathematics 1, Q 1(a); editorial change: include a preliminary part showing that we can represent complex numbers in the form $a\mathbf{I} + b\mathbf{J}$ (and then remove the definitions of **I** and **J** from the main question)

17 O&C, A level Mathematics (MEI), 1988, QP 9658/0, Further Mathematics 0 (Special Paper), Q 9; editorial change: explain what the term 'back substitution' means in a footnote.