THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Enrichment Programme for Young Mathematics Talents

Towards Differential Geometry

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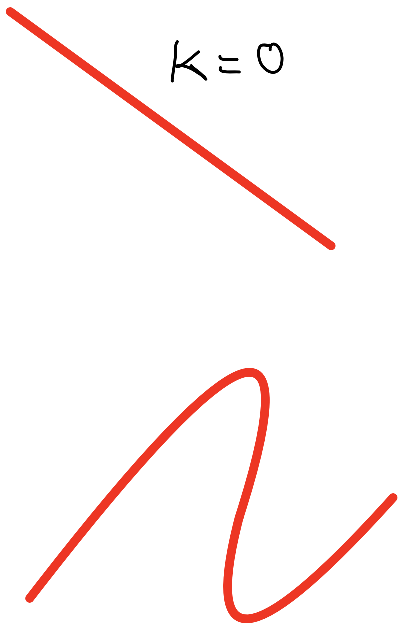
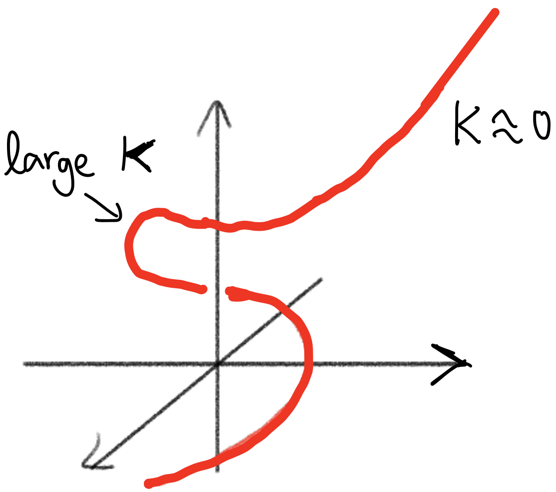
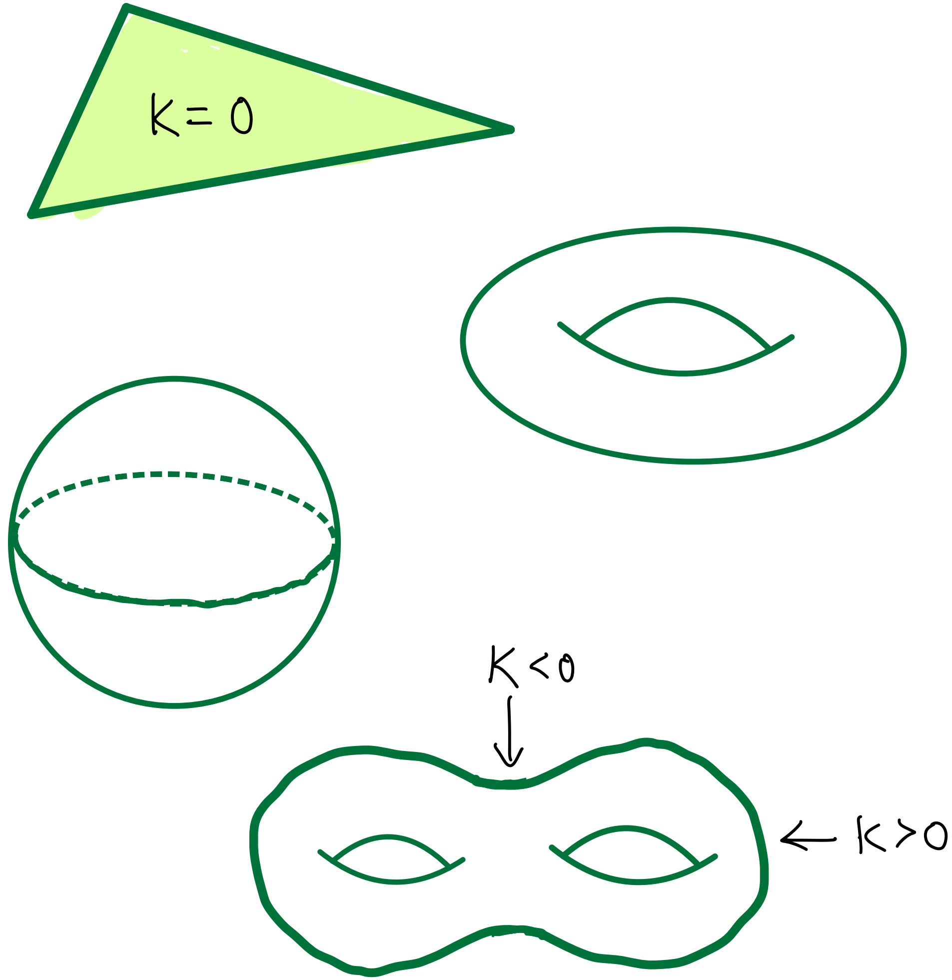
Objects to study



Plane Curves



Space Curve (in 3D)

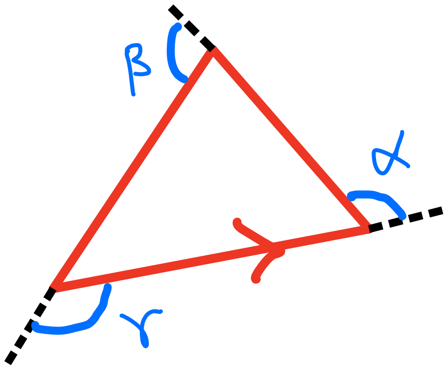


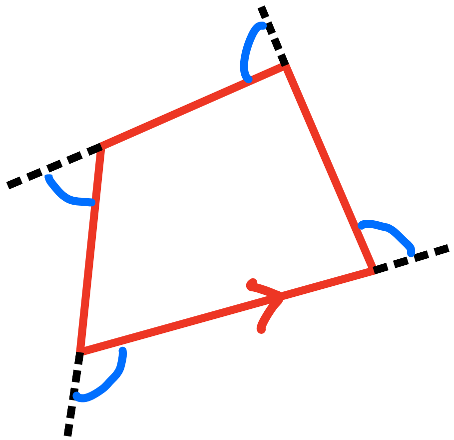
Surfaces (in 3D)

Study arclength, area, curvature…

κ

Tool: Calculus



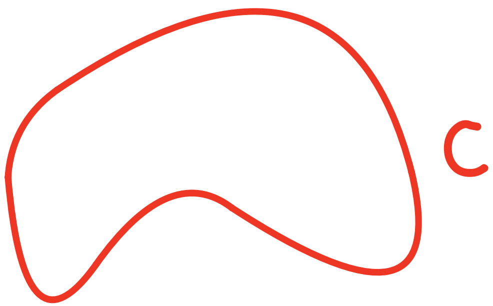


$\alpha + \beta + \gamma = 2\pi$

$Sum = 2\pi$

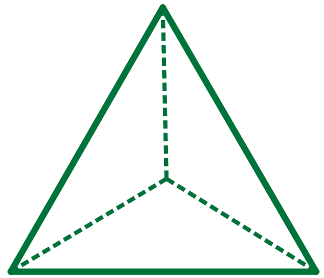
"Totally turns 2π"

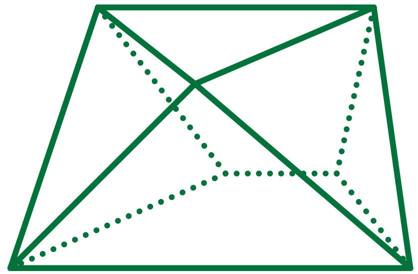
Continuous version



$$\int_C \kappa \, ds = 2\pi$$

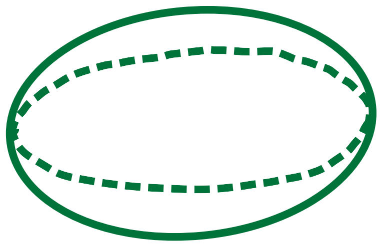
Similar phenomenon for surfaces

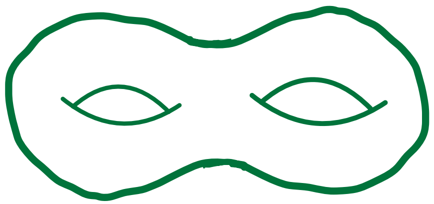




Sum of "angle defects" among all vertices $= 4\pi$

Gauss-Bonnet Theorem





$$\iint_S K dA = 4\pi$$

$$\iint_S K dA = -4\pi$$

1.1 Matrices

Definition 1.1.1 (Matrix). *Let m and n be positive integers. An $m \times n$* **matrix** *over $\mathbb{R}$ ($\mathbb{C}$) is a rectangular array of the form*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

m rows
n columns

where a_{ij}, $1 \le i \le m$, $1 \le j \le n$, are real (complex) numbers. We may also write a matrix as $A = [a_{ij}]$.

eg

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix} \qquad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2×3

$$\begin{array}{lll} a_{11} = 1 & a_{12} = 2 & a_{13} = 0 \\ a_{21} = -1 & a_{22} = 3 & a_{23} = 4 \end{array}$$

3×1

1. **Matrix addition**: *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then*

$$[A+B]_{ij} = a_{ij} + b_{ij}$$

In other words

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

eg

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix}$$

2. **Scalar multiplication**: *Let* $A = [a_{ij}]$ *be a* $m \times n$ *matrix and* c *be a real (complex) number. Then*

$$[cA]_{ij} = ca_{ij}$$

In other words

$$c\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}.$$

eg

$$(-2)\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -6 & -8 \\ -10 & -12 \end{bmatrix}$$

3. **Matrix multiplication**: *Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix. The matrix product of A and B is an $m \times p$ matrix and*

$$[AB]_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$$

for $1 \le i \le m$, $1 \le k \le p$. Note that the ik-th entry of AB is the sum of the products of the corresponding entries in the i-th row of A and the k-th column of B.

eg

$$\underset{2\times 2}{\overset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}}} \underset{2\times 3}{\overset{B}{\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}}} = \underset{2\times 3}{\overset{AB}{\begin{bmatrix} 4 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}}}$$

equal

$$[AB]_{11} = a_{11}b_{11} + a_{12}b_{21}$$

$$[AB]_{23} = a_{21}b_{13} + a_{22}b_{23}$$

$$BA = \underset{2\times 3}{\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}}$$

not equal

$\Rightarrow$ BA is not defined

eg

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$$

- (Diagonal matrix). *An $n \times n$ matrix of the form*

$$D = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

diagonal

eg

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- *The $m \times n$* **zero matrix** *is the matrix which every entry equals to 0.*

eg

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- *The* **identity matrix** *of size n is the matrix*

$$I_n = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

eg

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

For a $m \times n$ matrix A,

$$I_m A = A I_n = A$$

Remarks 1.1.4. *Let A, B, C be matrices.*

1. *AB is defined only when the number of columns of A is equal to the number of rows of B.*
2. *In general, $AB \neq BA$ even when they are both defined and of the same type.*
3. *In general, $AB = 0$ does not implies that $A = 0$ or $B = 0$.*
4. *In general, $AB = AC$ and $A \neq 0$ does not implies $B = C$.*

2.
$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

3.
$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but $A \neq 0$, $B \neq 0$

4.
$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{C}{\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = AB$$

but $B \neq C$

Compare: $2x = 2y \Rightarrow x = y$ Why?

Answer: Inverse exists

$$x = \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 2y = y$$

1. $(AB)C = A(BC)$ (Associative)

2. $(A+B)C = AC + BC$ *and* $C(A+B) = CA + CB$ (Distributive)

3. $c(AB) = (cA)B = A(cB)$

1.

$$\overbrace{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}}^{\text{first}} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}}^{\text{first}} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

<u>Remark</u> Associativity also holds for multiplying more than 3 matrices.

<u>eg</u> $$(A\,(BC))\,D = (AB)\,(CD)$$

Definition 1.1.6 (Transpose). *The* **transpose** *of an* $m \times n$ *matrix* $A = [a_{ij}]$ *is the* $n \times m$ *matrix* A^T *obtained by interchanging rows and columns of* A, *i.e.,*

$$[A^T]_{ji} = a_{ij}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$.

eg

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Proposition 1.1.7 (Properties of transpose). *Let* A *and* B *be matrices.*

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $(AB)^T = B^T A^T$

4. $\underset{m\times n}{A}\ \underset{n\times p}{B} \Rightarrow AB$ is $m\times p$ $\qquad \underset{p\times n}{B^T}\ \underset{n\times m}{A^T} \Rightarrow B^TA^T$ is $p\times m$

Definition 1.1.8 (Symmetric and anti-symmetric matrices). *Let A be an $n \times n$ matrix.*

1. *We say that A is a* **symmetric matrix** *if $A^T = A$.*
2. *We say that A is an* **anti-symmetric matrix** *(or a* **skew-symmetric matrix***) if $A^T = -A$.*

eg

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad A = A^T$$ Symmetric

$$B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \qquad B^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \qquad B^T = -B$$ Anti-symmetric

$$C = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & -2 \\ -1 & -2 & 4 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$$

Symmetric Anti-symmetric

Rmk. Diagonal entries of an anti-symmetric matrix are zero

Definition 1.1.12 (Matrix inverse). *An $n \times n$ matrix A is said to be* **invertible**, *if there exists a matrix A^{-1} called the* **inverse** *of A such that*

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- If $ad - bc = 0$, then A does not have an inverse.
- If $ad - bc \neq 0$, then A^{-1} exists.

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

eg

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{(1)(4) - (2)(3)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Proposition 1.1.13 (Properties of inverse). *Let A and B be two invertible $n \times n$ matrices over real (complex) numbers.*

1. *The inverse A^{-1} is invertible and $(A^{-1})^{-1} = A$*

2. *For any nonnegative integer k, A^k is invertible and $(A^k)^{-1} = (A^{-1})^k$. This allows us to define $A^{-k} = (A^{-1})^k$.*

3. *For any nonzero real (complex) number c, cA is invertible and*

$$(cA)^{-1} = c^{-1}A^{-1}$$

4. *The product AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

5. *A^T is invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

4. $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

5 $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

1.2 Determinant

Definition 1.2.1 (Determinant). *Let n be a positive integer and*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be an $n \times n$ matrix. The **determinant** *of A is denoted by*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and is defined inductively by

1. *For $n = 1$, we have $\det(A) = a_{11}$.*

2. *For $n > 1$, we have*

$$\det(A) = \underline{a_{11}} \det(A_{11}) - \underline{a_{12}} \det(A_{12}) + \cdots + (-1)^{n+1} \underline{a_{1n}} \det(A_{1n})$$

entries on the first row of A

where A_{ij}, $1 \leq i, j \leq n$ is the submatrix of A obtained by deleting the i-th row and the j-th column of A.

$$\boxed{\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n+1}a_{1n}\det(A_{1n})}$$

<u>$n = 2$</u>

$$\begin{aligned}\det A &= a_{11}\det(A_{11}) - a_{12}\det(A_{12})\\ &= a_{11}\det(a_{22}) - a_{12}\det(a_{21})\\ &= a_{11}a_{22} - a_{12}a_{21}\end{aligned}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$A_{11} = (a_{22})$ $\qquad$ $A_{21} = (a_{21})$

<u>eg.</u>

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

<u>$n = 3$</u>

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

<u>eg</u>

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (1)\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - (2)\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (3)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (1)(45-48) - (2)(36-42) + (3)(32-35)$$

$$= 0$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

$$\boxed{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}}$$

<u>eg</u>

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (1)(5)(9) + (2)(6)(7) + (3)(4)(8) - (3)(5)(7) - (2)(4)(9) - (1)(8)(6) = 0$$

$\backslash \quad \backslash \quad \backslash \quad - \quad / \quad / \quad /$

$\underline{n > 3}$

Wrong

eg
$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{vmatrix} = 0 + (1)\,(2)\,(3)\,(4) + 0 + 0 - 0 - 0 - 0 - 0 = 24$$

From definition:

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{vmatrix} = (0)\begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{vmatrix} - (1)\begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{vmatrix} + (0)\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{vmatrix} - (0)\begin{vmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= 0 - (1)\,(2)\,(3)\,(4) + 0 - 0$$
$$= -24$$

Proposition 1.2.4 (Direct formula for determinant). *Let n be a positive integer and $A = [a_{ij}]$. Then*

$$\det(A) = \sum_{\sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where S_n is the set of all permutations[1] *of $1, 2, \cdots, n$ and $sign(\sigma) = 1, -1$ when σ is a composition of even, odd number of transpositions*[2] *respectively.*

eg

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{aligned} &+a_{11}a_{22}a_{33}a_{34} - a_{12}a_{23}a_{34}a_{41} \\ &+a_{13}a_{24}a_{31}a_{42} - a_{14}a_{21}a_{32}a_{43} \\ &-a_{12}a_{21}a_{33}a_{44} + \ldots . \end{aligned}$$

totally 24 terms

Proposition 1.2.6 (Determinant under row and column operations). *Let A be an $n \times n$ matrix.*

1. *If B is obtained from A by multiplying a single row (or column) of A by a constant k, then $\det(B) = k\det(A)$.*
2. *If B is obtained from A by interchanging two rows (or columns) of A, then $\det(B) = -\det(A)$.*
3. *If B is obtained from A by adding a constant multiple of one row (or column) of A to another row (or column) of A, then $\det(B) = \det(A)$.*

eg

$$\begin{vmatrix} 10 & 2 & 3 \\ 10 & 2 & 1 \\ 20 & 3 & 0 \end{vmatrix} \underset{1}{=} 10 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 0 \end{vmatrix} \underset{3}{=} 10 \begin{vmatrix} 0 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= 10\,[(2)\,(1)\,(3) - (2)\,(2)\,(2)]$$
$$= -20$$

Proposition 1.2.8 (Further properties of determinant). *Let A be an $n \times n$ matrix.*

1. *If A has a row (or column) consisting entirely of zeros, then* $\det(A) = 0$.
2. *If two rows (or columns) of A are identical, then* $\det(A) = 0$.
3. *If A is an upper triangular matrix, that is,*

$$A = \begin{pmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ \mathit{0} & & & a_{nn} \end{pmatrix}$$

   Then $\det(A) = a_{11}a_{22}\cdots a_{nn}$. *In particular, the determinant of a diagonal matrix is the product of its diagonal entries.*

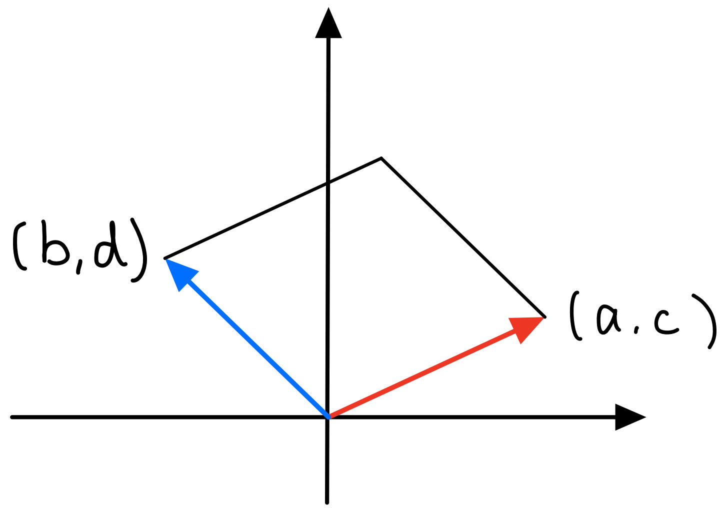
4. $\det(cA) = c^n \det(A)$ *for any* $c \in \mathbb{R}$. *(Caution!* $\det(cA) \neq c\det(A)$*)*
5. $\det(A^T) = \det(A)$

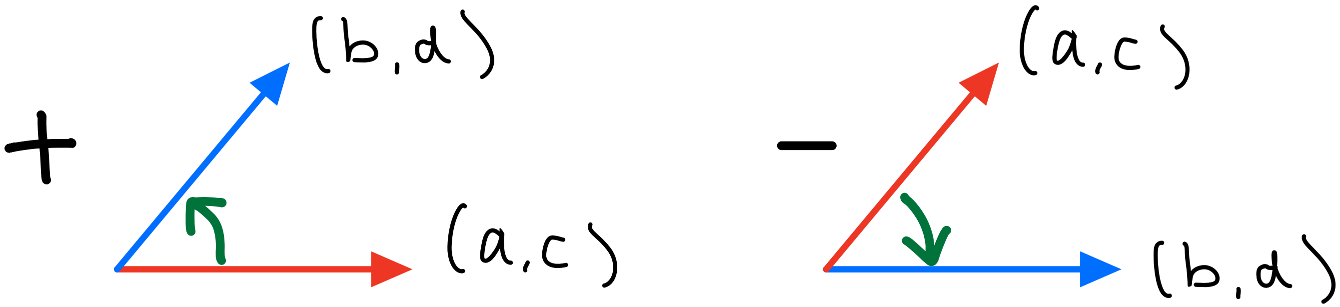
4. $$\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix} = 3\begin{vmatrix} a & b \\ 3c & 3d \end{vmatrix} = (3)^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Geometric interpretation of determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm \text{Area of the parallelogram}$$

sign determined by orientation





<u>Rmk</u> Similar interpretation for considering row vectors (a,b) and (c,d).

Proposition 1.2.9. *Let A and B be two $n \times n$ matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the (i, j) **cofactor** by

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

where M_{ij} is the submatrix of A obtained by deleting the i-th row and j-th column of A.

Define the adjugate matrix of A, $\operatorname{adj}(A)$, by $[\operatorname{adj}(A)]_{ij} = A_{ji}$

Proposition 1.2.11. *Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then A is invertible, that is, the inverse A^{-1} of A exists, if and only if $\det(A) \neq 0$. Moreover if A is invertible, then*

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

where $\operatorname{adj}(A)$ is the adjugate matrix of A.

eg. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A_{11} = d \quad A_{12} = -c$
$\quad A_{21} = -b \quad A_{22} = a$

$$adj\,(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

eg

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{array}{ll} A_{ij} = (-1)^{i+j} \det(M_{ij}) & \\ & A^{-1} = \dfrac{1}{\det(A)} \mathrm{adj}(A) \\ [\mathrm{adj}(A)]_{ij} = A_{ji} & \end{array}}$$

$$adj\,(A) = \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -2 \\ -2 & -1 & 4 \\ 1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -2 & 4 & 1 \end{bmatrix}$$

$$\det A = (1)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - (2)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = 1 + 4 - 2 = 3$$

$$A^{-1} = \frac{1}{\det A} adj\,(A) = \frac{1}{3} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -2 & 4 & 1 \end{bmatrix}$$

Definition 1.2.12 (Trace)**.** *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The* **trace** *of A is defined by*

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

$$\mathrm{tr}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 + 5 + 9 = 15 \qquad \mathrm{tr}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 + 4$$

Proposition 1.2.13 (Properties of trace)**.** *Let A, B be $n \times n$ matrices and $k \in \mathbb{R}$. Then*

1. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
2. $\mathrm{tr}(kA) = k\mathrm{tr}(A)$
3. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

1.3 Vectors

Definition 1.3.1 (Euclidean space)**.** *Let n be a positive integer. The n dimensional* **Euclidean space** *is the set*

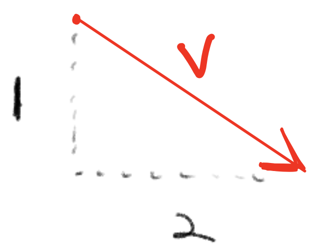
$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for any } i = 1, 2, \ldots, n\}.$$

A n-dimensional vector $\mathbf{v}$ is an element of $\mathbb{R}^n$.

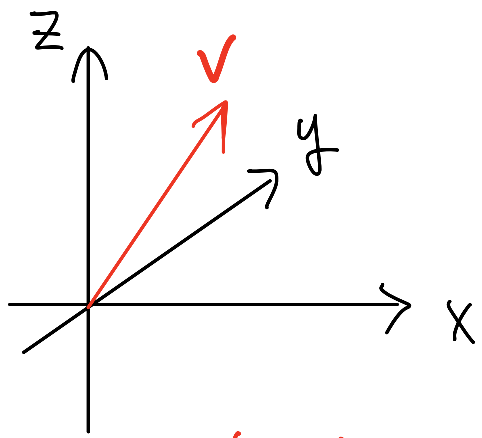
Notations $\mathbf{v}$, $\overrightarrow{v}$, $\overline{v}$, v

e.g. $V = (2, -1)$ in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

$V = (1, 2, 3)$ in $\mathbb{R}^3$



V, $\vec{V}$, $\bar{V}$,



DSE: $V = 2i - j$

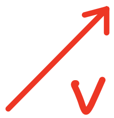
$V = i + 2j + 3k$

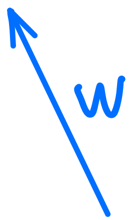
Definition 1.3.2 (Vector addition and scaler multiplication).

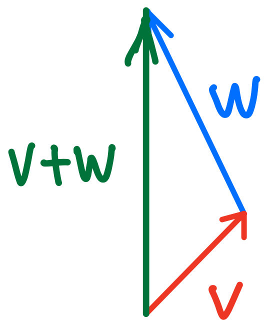
1. **Vector addition**: *Let* $\mathbf{u} = (u_1, u_2, \ldots, u_n), \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. *Define*
$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

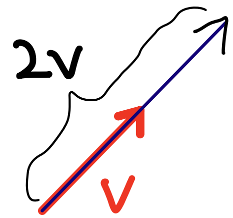
2. **Scalar multiplication**: *Let* $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$. *Define*
$$\alpha\mathbf{v} = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n).$$

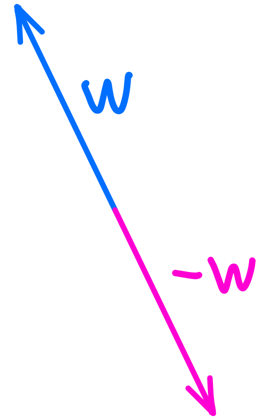
e.g. $\vec{v} = (1, 1)$ $\vec{w} = (-1, 2)$









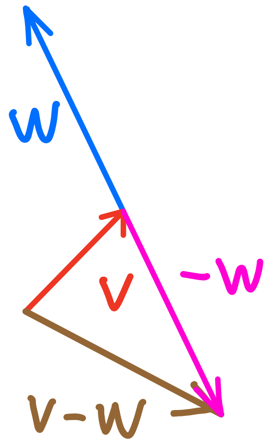


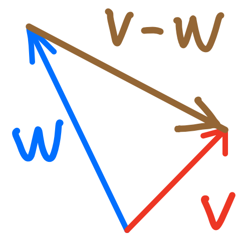
$\vec{v} + \vec{w} = (0, 3)$

$2\vec{v} = (2, 2)$

$-\vec{w} = (1, -2)$

$\vec{v} - \vec{w} = (2, -1)$





Definition 1.3.3 (Scalar product)**.** *Let* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$,$\mathbf{v}$= $(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. *The* **scalar product**, *or* **dot product**, *of* $\mathbf{u}$ *and* $\mathbf{v}$ *is defined by*

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Note that the scalar product of two vectors is a number, not a vector.

eg $\langle (1,2,3), (4,5,6) \rangle = (1)(4) + (2)(5) + (3)(6) = 32$

Rmk $\mathbf{u} \bullet \mathbf{v}$ is another common notation for scalar product

Proposition 1.3.4 (Properties of scalar product)**.** *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$. *Then*

1. *(Bilinear):*
$$\langle \alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$$

2. *(Symmetric):*
$$\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

3. *(Positive definite):*
$$\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$$
with equality holds if and only if $\mathbf{v} = \mathbf{0}$.

Definition 1.3.6 (Norm). *Let* $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. *The* **norm** *of* $\mathbf{v}$ *is defined as*

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

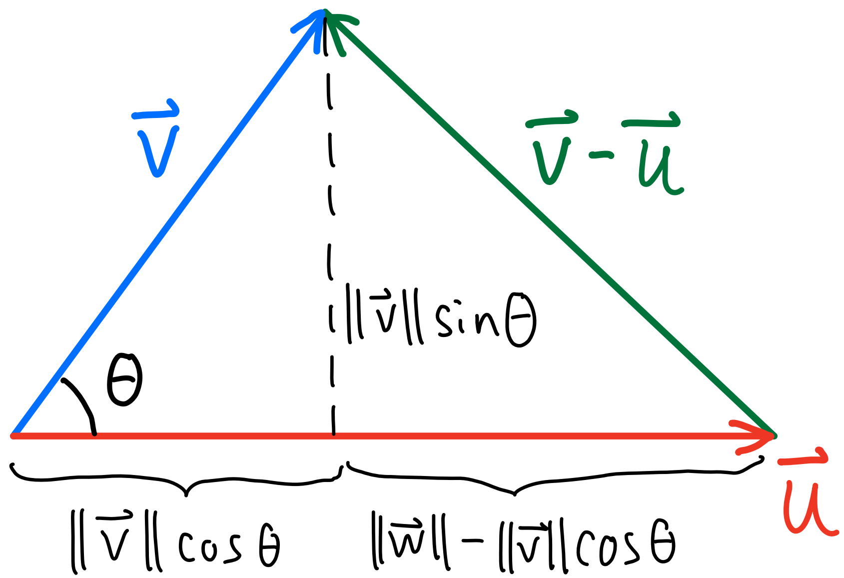
norm = length

Generalize length to dimension ≥ 4

Definition 1.3.7 (Unit vector). *We say that* $\mathbf{v} \in \mathbb{R}^n$ *is a* **unit vector** *if* $\|\mathbf{v}\| = 1$.

Proposition $\langle u, v \rangle = \|u\| \, \|v\| \cos\theta$, where θ is the angle between u, v in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Pf



① $\|v-u\|^2 = (\|v\|\sin\theta)^2 + (\|u\| - \|v\|\cos\theta)^2$

$= \|v\|^2 \sin^2\theta + \|u\|^2 - 2\|u\|\|v\|\cos\theta + \|v\|^2\cos^2\theta$

$= \underline{\|v\|^2 + \|u\|^2} - 2\,\underline{\|u\|\|v\|\cos\theta}$

② $\|v-u\|^2 = \langle v-u, v-u \rangle$

$= \langle v,v \rangle - \langle u,v \rangle - \langle v,u \rangle + \langle u,u \rangle$

$= \underline{\|v\|^2 + \|u\|^2} - 2\,\underline{\langle u,v \rangle}$

Theorem 1.3.8 (Cauchy-Schwarz inequality). *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *we have*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

with equality holds if and only if $\mathbf{u} = \mathbf{0}$ *or* $\mathbf{v} = \alpha \mathbf{u}$ *for some real number* α.

$$-\|u\| \|v\| \leqslant \langle u, v \rangle \leqslant \|u\| \|v\| \Rightarrow -1 \leqslant \frac{\langle u, v \rangle}{\|u\| \|v\|} \leqslant 1$$

Definition 1.3.10 (Angle between two vectors). *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *be two nonzero vectors. The* **angle** *between* $\mathbf{u}$ *and* $\mathbf{v}$ *is the unique* $\theta \in [0, \pi]$ *such that*

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

Generalize angle to dimension ≥ 4

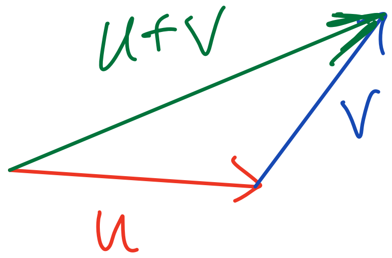
Definition 1.3.11 (Orthogonal vectors). *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *be two vectors. We say that* $\mathbf{u}$ *and* $\mathbf{v}$ *are* **orthogonal** *and write* $\mathbf{u} \perp \mathbf{v}$ *if*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

orthogonal = perpendicular

Proposition 1.3.9 (Properties of norm). *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.*

1. $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$
2. $\|\mathbf{v}\| \geq 0$ *with* $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$.
3. *(Triangle inequality):*
$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$
4. *(Parallelogram law):*
$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4} \left(\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 \right)$$



Pf 3

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v \rangle \\ &= \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle \\ &\leq \|u\|^2 + 2\,|\langle u, v \rangle| + \|v\|^2 \\ &\leq \|u\|^2 + 2\,\|u\|\,\|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

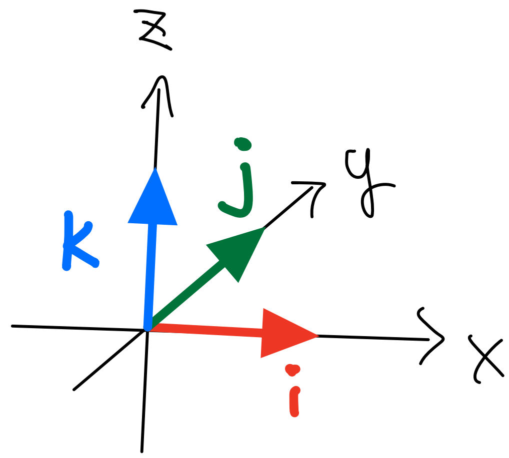
Definition 1.3.12 (Cross product). *Let* $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ *be two vectors in* $\mathbb{R}^3$. *The* **cross product**, *or* **vector product**, *of* $\mathbf{u}$ *and* $\mathbf{v}$ *is defined by*

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$
$$= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

where $\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)$.

eg $\quad u = (2,3,5) \quad v = (1,2,3)$

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} i - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} j + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} k$$
$$= (-1)\, i - (1)\, j + (1) k$$
$$= (-1,0,0) - (0,1,0) + (0,0,1)$$
$$= (-1,-1,1)$$



$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

Proposition 1.3.13 (Properties of cross product).

1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
2. *(Bilinear) For any* $\alpha, \beta \in \mathbb{R}$ *and* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$,
$$(\alpha \mathbf{u} + \beta \mathbf{v}) \times \mathbf{w} = \alpha \mathbf{u} \times \mathbf{w} + \beta \mathbf{v} \times \mathbf{w}$$
3. *(Anti-symmetric) For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,
$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$
4. *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, *we have* $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ *and* $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$, *that is*
$$\underbrace{\langle \mathbf{u} \times \mathbf{v}, \mathbf{u} \rangle}_{\begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}} = \langle \mathbf{u} \times \mathbf{v}, \mathbf{v} \rangle = 0$$

$$v \times u = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

5. *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

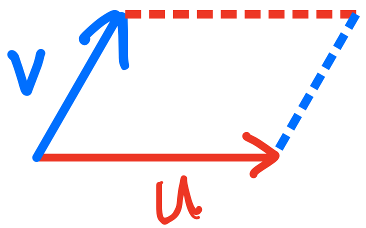
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta \, \mathbf{n}$$

where θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$, *and* $\mathbf{n}$ *is the unit vector normal to the plane spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ *with direction determined by the right hand rule. In other words,*

(a) $\|\mathbf{u} \times \mathbf{v}\|$ *is equal to the area of the parallelogram spanned by* $\mathbf{u}$, $\mathbf{v}$.

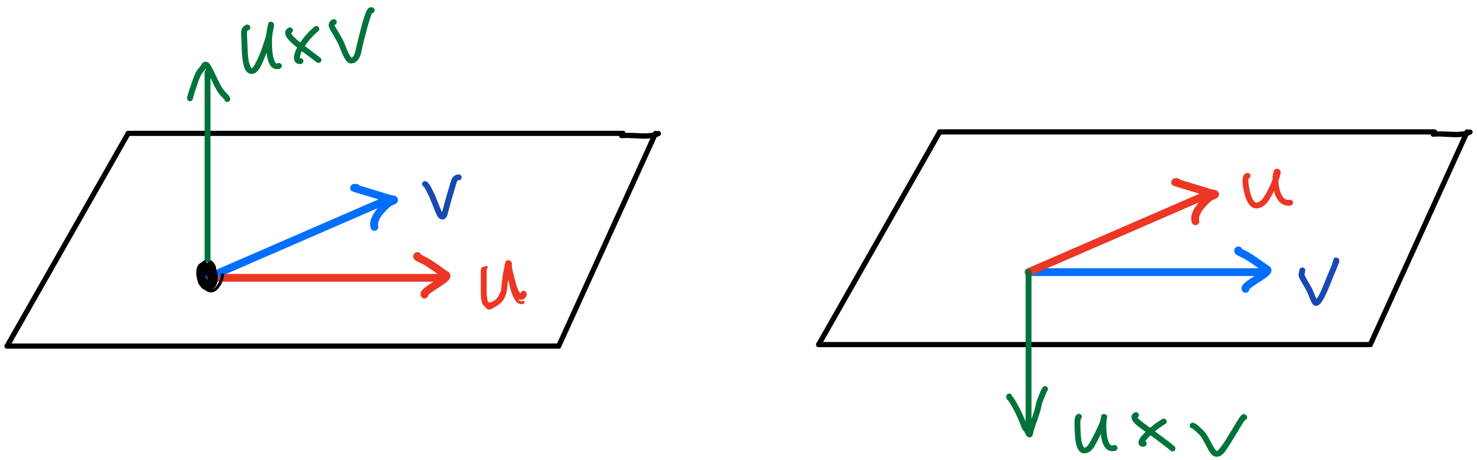
(b) $\mathbf{u} \times \mathbf{v}$ *is normal to the plane spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ *with direction determined by the right hand rule.*

(a)



$$Area = \|u \times v\|$$

(b)



Definition 1.3.14 (Scalar triple product). *Let* $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$. *The* **scalar triple product** *of* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *is defined by*

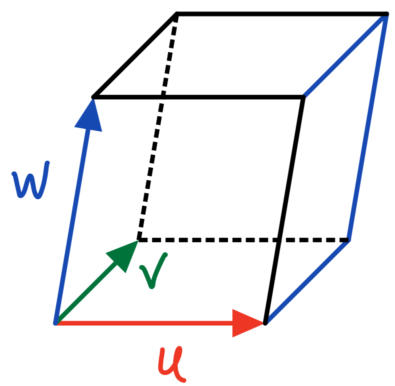
$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle \underset{*}{=} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

$$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\begin{aligned} * \quad L.H.S. &= \langle u, v \times w \rangle \\ &= \left\langle (u_1, u_2, u_3), \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \right\rangle \\ &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = R.H.S. \end{aligned}$$

<u>Rmk</u>

$\langle u, v \times w \rangle = \pm$ volume of



Proposition 1.3.16 (Cyclic property of scalar triple product)**.** *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *be three vectors. We have*

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \times \mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{u} \times \mathbf{v} \rangle$$

The following three identities are useful in studying curvature of surfaces.

Proposition 1.3.17.

1. *For any* $\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2 \in \mathbb{R}^3$,

$$\begin{vmatrix} \langle \mathbf{u}_1, \mathbf{u}_2 \rangle & \langle \mathbf{u}_1, \mathbf{v}_2 \rangle \\ \langle \mathbf{v}_1, \mathbf{u}_2 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \end{vmatrix} = \langle \mathbf{u}_1 \times \mathbf{v}_1, \mathbf{u}_2 \times \mathbf{v}_2 \rangle$$

2. *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\begin{vmatrix} \langle \mathbf{u}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{v} \rangle \\ \langle \mathbf{v}, \mathbf{u} \rangle & \langle \mathbf{v}, \mathbf{v} \rangle \end{vmatrix} = \|\mathbf{u} \times \mathbf{v}\|^2$$

3. *For any* $\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{21}, \mathbf{x}_{22}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\begin{aligned} &\begin{vmatrix} \langle \mathbf{x}_{11}, \mathbf{u} \times \mathbf{v} \rangle & \langle \mathbf{x}_{12}, \mathbf{u} \times \mathbf{v} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{u} \times \mathbf{v} \rangle & \langle \mathbf{x}_{22}, \mathbf{u} \times \mathbf{v} \rangle \end{vmatrix} \\ = &\begin{vmatrix} \langle \mathbf{x}_{11}, \mathbf{x}_{22} \rangle - \langle \mathbf{x}_{12}, \mathbf{x}_{21} \rangle & \langle \mathbf{x}_{11}, \mathbf{u} \rangle & \langle \mathbf{x}_{11}, \mathbf{v} \rangle \\ \langle \mathbf{x}_{22}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{v} \rangle \\ \langle \mathbf{x}_{22}, \mathbf{v} \rangle & \langle \mathbf{v}, \mathbf{u} \rangle & \langle \mathbf{v}, \mathbf{v} \rangle \end{vmatrix} \\ &- \begin{vmatrix} 0 & \langle \mathbf{x}_{12}, \mathbf{u} \rangle & \langle \mathbf{x}_{12}, \mathbf{v} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{v} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{v} \rangle & \langle \mathbf{v}, \mathbf{u} \rangle & \langle \mathbf{v}, \mathbf{v} \rangle \end{vmatrix} \end{aligned}$$

Vector-valued Functions

		output
Real-valued function	$f(t) = 2t + 1$	$\mathbb{R}$
Vector-valued function	$u(t) = (t, t^2)$	$\mathbb{R}^2$
	$w(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$	$\mathbb{R}^3$

Derivative

$$\mathbf{v}'(t) = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$$

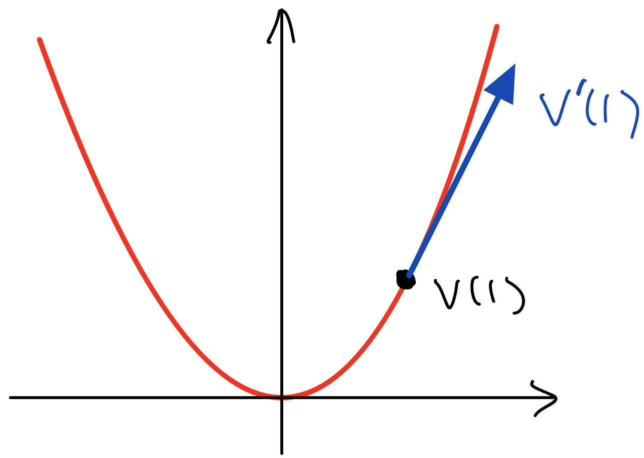
If $\mathbf{v}(t) = (x(t), y(t))$

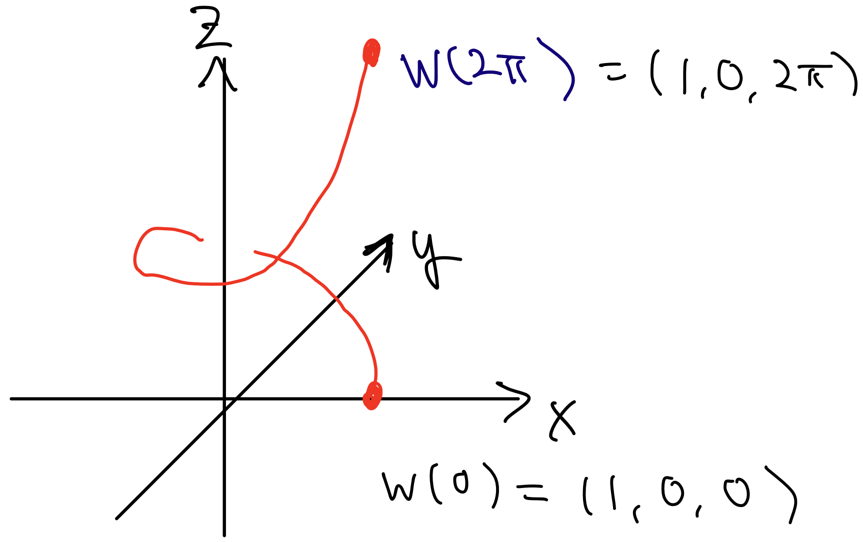
then $\mathbf{v}'(t) = (x'(t), y'(t))$

eg $v(t) = (t, t^2)$ $\quad v(1) = (1, 1)$

$v'(t) = (1, 2t)$ $\quad v'(1) = (1, 2)$

$$\begin{cases} x = t \\ y = t^2 \end{cases} \Rightarrow y = x^2$$





Proposition 1.3.34 (Rules for derivative of vector valued functions). *Let* $\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t)$ *be differentiable vector valued functions and* $\alpha(t)$ *be real valued function.*

1. $\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$
2. $\frac{d}{dt}(\alpha\mathbf{v}) = \alpha\frac{d\mathbf{v}}{dt} + \frac{d\alpha}{dt}\mathbf{v}$
3. $\frac{d}{dt}\langle\mathbf{u},\mathbf{v}\rangle = \langle\frac{d\mathbf{u}}{dt},\mathbf{v}\rangle + \langle\mathbf{u},\frac{d\mathbf{v}}{dt}\rangle$
4. $\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \frac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\frac{d\mathbf{v}}{dt}$
5. $\frac{d}{dt}\langle\mathbf{u},\mathbf{v}\times\mathbf{w}\rangle = \langle\frac{d\mathbf{u}}{dt},\mathbf{v}\times\mathbf{w}\rangle + \langle\mathbf{u},\frac{d\mathbf{v}}{dt}\times\mathbf{w}\rangle + \langle\mathbf{u},\mathbf{v}\times\frac{d\mathbf{w}}{dt}\rangle$

eg

$$\frac{d}{dt}\left(\left(1,t,t^2\right)\times(t,2t,3t)\right)$$

$$= \left(\frac{d}{dt}\left(1,t,t^2\right)\right)\times(t,2t,3t) + \left(1,t,t^2\right)\times\left(\frac{d}{dt}(t,2t,3t)\right)$$

$$= (0,1,2t)\times(t,2t,3t) + \left(1,t,t^2\right)\times(1,2,3)$$

Lemma 1.3.35. *Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be two vector valued functions.*

1. *If $\langle \mathbf{u}(t), \mathbf{v}(t)\rangle$ is constant, then for any t, we have*

$$\langle \mathbf{u}'(t), \mathbf{v}(t)\rangle = -\langle \mathbf{u}(t), \mathbf{v}'(t)\rangle.$$

2. *If $\|\mathbf{v}(t)\|$ is constant, then for any t, we have*

$$\langle \mathbf{v}'(t), \mathbf{v}(t)\rangle = 0.$$

Pf ① $\langle u(t), v(t)\rangle = C$ constant

$\frac{d}{dt}\downarrow$

$\langle u'(t), v(t)\rangle + \langle u(t), v'(t)\rangle = 0$

② $\|v(t)\|^2 = \langle v(t), v(t)\rangle$ is constant

By ① $\langle v'(t), v(t)\rangle = -\langle v(t), v'(t)\rangle$

$\Rightarrow \langle v'(t), v(t)\rangle = 0$