Homework 1

Solutions

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(1.1) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the curve

$$\gamma(t) = (t, \sinh t, \cosh t).$$

a) Calculate the arc-length $L\left(\gamma|_{[0,\frac{1}{2}\log(2)]}\right)$.

b) Calculate its curvature κ.

c) Calculate its torsion τ.

Solution (1.1) We will use the following formulas throughout this question:

$$\begin{aligned}\gamma'(t) &= (1, \cosh t, \sinh t),\\ \gamma''(t) &= (0, \sinh t, \cosh t),\\ \gamma'''(t) &= (0, \cosh t, \sinh t).\end{aligned}$$

a)

$$\|\gamma'(t)\| = \sqrt{1 + \cosh^2 t + \sinh^2 t} = \sqrt{2}\cosh(t).$$

Thus

$$\begin{aligned}L\left(\gamma|_{[0,\frac{1}{2}\log(2)]}\right) &= \int_0^{\frac{1}{2}\log(2)} \sqrt{2}\cosh s\, ds\\ &= \left(\sqrt{2}\sinh s\right|_0^{\log(\sqrt{2})}\\ &= \sqrt{2}\sinh(\log(\sqrt{2})) = \frac{1}{2}.\end{aligned}$$

b) Since

$$\gamma'(t) \times \gamma''(t) = (1, -\cosh t, \sinh t),$$

it follows that

$$\|\gamma'(t)\| = \|\gamma'(t) \times \gamma''(t)\| = \sqrt{2}\cosh(t),$$

and therefore, the curvature is given by the equation

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{1}{2\cosh^2(t)}.$$

c) Since

$$\langle \gamma'(t) \times \gamma''(t), \gamma'''(t)\rangle = (1, -\cosh t, \sinh t) \cdot (0, \cosh t, \sinh t) = -1,$$

the torsion is given by the equation

$$\tau(t) = \frac{\langle \gamma'(t) \times \gamma''(t), \gamma'''(t)\rangle}{\|\gamma'(t) \times \gamma''(t)\|^2} = -\frac{1}{2\cosh^2(t)}.$$

(1.2) Let $u : I \to \mathbb{R}$ be a smooth function, and consider the smooth curve $\gamma_u : I \to \mathbb{R}^2$ defined by

$$\gamma_u(t) = (t, u(t)), \quad \forall t \in I.$$

a) Show that γ_u is a regular curve and describe its trace in terms of the function u.

b) Calculate the curvature of γ_u in terms of the derivatives of u.

c) Choosing $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $u(t) := -\log(\cos(t))$, sketch the trace of γ_u. What is the supremum of the curvature for this example?

Solution (1.2)

a) $\|\gamma_u'(t)\| = \sqrt{1 + u'(t)^2} \geq 1$, and the trace of γ_u is the graph of u.

b) Using the formula for curvature we have

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{|u''|}{(1 + (u')^2)^{\frac{3}{2}}}.$$

c) The trace is the grim reaper curve, and its curvature is given by $\cos(t)$, which has a supremum of 1 at $t = 0$.

(1.3) We take the unit sphere to be the regular surface

$$\mathbb{S}^2 := \{v \in \mathbb{R}^3 : \|v\| = 1\}.$$

Suppose $\gamma : \mathbb{R} \to \mathbb{S}^2$ is a smooth regular curve lying within the unit sphere. Show that the curvature of γ is non-zero everywhere.

Hint: Taylor's theorem

Solution (1.3)

Solution 1: Parameterise by arc-length. Assume that the curvature vanishes, i.e $\gamma''(s) = 0$. Differentiating the equality $\|\gamma(s)\|^2 = 1$ twice yields $\|\gamma'(s)\| = 0$, which is an obvious contradiction.

Solution 2: Without loss of generality, it suffices to show γ has non-zero curvature at $s = 0$. By Taylor's theorem, for s sufficiently small, we have

$$\gamma(s) = \gamma(0) + s\gamma'(0) + s^2\gamma''(0) + O(s^3).$$

If the curvature vanishes, then $\gamma''(0) = 0$, and taking the length of both sides yields

$$1 = \|\gamma(s)\|^2 = \|\gamma(0) + s\gamma'(0)\|^2 + O(s^3) = 1 + s^2\|\gamma'(0)\| + O(s^3),$$

where the final equality follows from $\gamma \perp \gamma'$. This implies that $\|\gamma'(0)\| = 0$, but this is a contradiction to γ being regular.

(1.4) Suppose $\gamma : \mathbb{R} \to \mathbb{S}^2$ is a smooth regular curve parameterised by arc-length lying within the unit sphere. By the previous question, its curvature is everywhere non-zero, and so expressing $\gamma(s)$ with respect to its Frenet frame, we find the smooth functions $a, b, c : I \to \mathbb{R}$ such that

$$\gamma(s) = a(s)T(s) + b(s)N(s) + c(s)B(s), \quad \forall s \in I.$$

a) Show that $a \equiv 0$.

b) By differentiating and using the Frenet formulas, show that $b = -\kappa^{-1}$.

c) In the case where the torsion is non-zero, find a formula for c in terms of curvature and torsion.

d) Show that κ and τ solve the differential equation

$$(\kappa')^2 = \kappa^2(\kappa^2 - 1)\tau^2.$$

Solution (1.4)

a) As γ is on the sphere, $\gamma \perp \gamma' = T$, and so $a \equiv 0$.

b) Differentiating yields

$$T = \gamma' = b'N + b(-\kappa T + \tau B) + c'B + c(-\tau N).$$

Equating coefficients, we find that

$$1 = -\kappa b, \quad b' - \tau c = 0, \quad \tau b + c' = 0.$$

The first equation is exactly what we needed to show.

c) If the torsion is non-zero, then from the second equation above we have

$$c = \frac{b'}{\tau} = \frac{\kappa'}{\kappa^2 \tau}.$$

d) We first deal with the case $\tau = 0$. From the notes, we know that γ must then be a plane curve, which combined with lying on the unit sphere, means that γ is a circular arc, and hence has constant curvature. This trivially solves the differential equation.

Therefore, we may assume the torsion is non-zero. By parts a)-c), we find that

$$\gamma = -\kappa^{-1}N + \frac{\kappa'}{\kappa^2\tau}B.$$

Since $N \perp B$, taking the length of both sides gives

$$1 = \frac{1}{\kappa^2} + \frac{(\kappa')^2}{\kappa^4\tau^2}.$$

Rearranging, we conclude that

$$(\kappa')^2 = \kappa^4\tau^2 - \kappa^2\tau^2 = \kappa^2(\kappa^2 - 1)\tau^2.$$