Midterm Practise Questions

(Q1)

a) Let $I \subseteq \mathbb{R}$ be an open interval and let $\gamma : I \to \mathbb{R}^3$ be a smooth regular curve. Define what it means for γ to be parameterised by arc-length.

b) Give the definition of the curvature κ_γ of γ at a point $s \in I$.

c) Suppose now that the trace $\gamma(I)$ is contained within the closed unit ball

$$\gamma(s) \in \overline{B} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}, \quad \forall s \in I.$$

If $\|\gamma(s_0)\| = 1$ for some $s_0 \in I$, show that the curvature satisfies

$$\kappa_\gamma(s_0) \geq 1.$$

d) Find the curvature of the ellipse $\{x^2 + 4y^2 = 1, z = 0\}$ where it touches the unit sphere.

e) Consider now a smooth regular curve $\eta : I \to \mathbb{R}^3$ contained with the cylinder

$$\eta(s) \in C := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\}, \quad \forall s \in I.$$

If $\eta(s_0)$ lies on the boundary of the cylinder

$$\eta(s_0) \in \partial C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\},$$

for some $s_0 \in I$, is it necessarily true that its curvature satisfies $\kappa_\eta(s_0) \geq 1$? Justify your answer.

Solution (1)

a) If $\|\gamma'(s)\| = 1$ for every $s \in I$.

b) If γ is parameterised by arc-length, then $\kappa_\gamma = \|\gamma''(s)\|$.

c) The smooth function $\|\gamma\|^2$ has a local maximum at s_0, and hence

$$0 \geq \frac{d^2}{ds^2}\|\gamma\|^2|_{s_0} = 2\gamma(s_0) \cdot \gamma''(s_0) + 2\|\gamma'(s_0)\|^2$$

Rearranging, and using an arc-length parameterisation,

$$-\gamma(s_0) \cdot \gamma''(s_0) \geq 1,$$

and by Cauchy-Schwarz we find

$$\kappa_\gamma(s_0) = \|\gamma''(s_0)\| \cdot \underbrace{\|-\gamma(s_0)\|}_{=1} \geq -\gamma(s_0) \cdot \gamma''(s_0) \geq 1.$$

d) We first parameterise the ellipse by

$$\gamma(t) = (\cos t, \frac{1}{2}\sin t, 0).$$

We will use the formula $\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}$ from the notes. Note that $\gamma'(t) = (-\sin t, \frac{1}{2}\cos t, 0)$, and $\gamma''(t) = (-\cos t, -\frac{1}{2}\sin t, 0)$. Thus, $\gamma' \times \gamma'' = (0, 0, \frac{1}{2})$, and $\|\gamma'\|^2 = \sin^2(t) + \frac{1}{4}\cos^2 t$. Plugging everything into our formula, we find

$$\kappa = \frac{1}{2(\sin^2(t) + \frac{1}{4}\cos^2 t)^{\frac{3}{2}}}.$$

Note that the ellipse touches the sphere precisely when $y = 0$, or $\sin(t) = 0$. Therefore, the curvature at these points is 4.

e) No. Consider the vertical line $\eta(t) = (1, 0, t)$. This curve is completely contained within the boundary ∂C, however it has zero curvature everywhere.

(Q2) Let $S \subseteq \mathbb{R}^3$.

a) Define what it means for S to be a regular surface.

b) Is the subset

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^3\}$$

a regular surface? Justify your answer.

c) Fix S to be a regular surface. Define what is means for a function $f : S \to \mathbb{R}$ to be smooth.

d) Consider the smooth function $f : S \to \mathbb{R}$ defined by

$$f(p) := \|p\|^2, \quad \forall p \in S.$$

Show that $p \in S$ is a critical point of f, if and only if the vector p is perpendicular to the tangent space T_pS.

e) Suppose now that S is also oriented with Gauss map $N : S \to \mathbb{S}^2$. Consider the smooth function $g : S \to \mathbb{R}$ defined by

$$g(p) = p \cdot N_p, \quad \forall p \in S.$$

Show that $p \in S$ is a critical point of g if and only if the vector p is perpendicular to the image of the shape operator at p.

Solution (2)

a) S is a regular surface if, for every $p \in S$, there exists open sets $U \subseteq \mathbb{R}^2$ and $p \in V \subseteq \mathbb{R}^3$, and a smooth map $X : U \to V \cap S \subseteq \mathbb{R}^3$, such that

(i) X is an immersion:

$$dX(q) : \mathbb{R}^2 \to \mathbb{R}^3 \text{ is injective, for all } q \in U.$$

(ii) X is a homeomorphism:

$$X \text{ is bijective with both } X \text{ and } X^{-1} \text{ continuous.}$$

b) No. If it was, then we showed in the course that it would locally be a smooth graph in a neighbourhood of the origin. Since the projection maps onto the $\{x = 0\}$ and $\{y = 0\}$ planes are not injective on any neighbourhood of the origin, it must a local graph over the $\{z = 0\}$ plane. However, the function $z = (x^2 + y^2)^{\frac{1}{3}}$ is NOT smooth at $x = y = 0$:

$$\lim_{x \downarrow 0} \frac{\partial z}{\partial x}(x, 0) = \lim_{x \downarrow 0} \frac{2}{3} x^{-\frac{1}{3}} = \infty.$$

c) f is said to be smooth at $p \in S$ if, for some coordinate chart $X : U \subseteq \mathbb{R}^2 \to V \cap S \subseteq \mathbb{R}^3$, with $p \in V$, the composition $f \circ X : U \to \mathbb{R}$ to smooth at $X^{-1}(p)$. We say that f is smooth if f is smooth at every $p \in S$.

d) For any smooth curve γ in S with $\gamma(0) = p$, we have

$$\begin{aligned} df_p \cdot \gamma'(0) &= (f \circ \gamma)'(0) \\ &= \frac{d}{dt} \|\gamma(t)\|^2 |_{t=0} \\ &= 2\gamma(0) \cdot \gamma'(0) \\ &= 2p \cdot \gamma'(0). \end{aligned}$$

Therefore, $df_p \cdot v = 0$ iff $p \cdot v = 0$ for any tangent vector $v \in T_pS$. It follows that p is a critical point of f if and only if $df_p \cdot v = 0$ for every $v \in T_pS$, iff $p \perp T_pS$.

e) Again, let γ be a smooth curve in S with $\gamma(0) = p$ and $\gamma'(0) = v \in T_pS$. Then

$$\begin{aligned} dg_p \cdot v &= (g \circ \gamma)'(0) \\ &= \frac{d}{dt} \langle \gamma(t), N_{\gamma(t)} \rangle |_{t=0} \\ &= \underbrace{\langle v, N_p \rangle}_{=0} + \langle p, dN_p \cdot v \rangle \\ &= \langle p, dN_p \cdot v \rangle . \end{aligned}$$

Therefore, p is a critical point of g iff $dN_p \cdot v$ is perpendicular to p for every $v \in T_pS$ iff p is perpendicular to the image of the shape operator $p \perp (-dN_p(T_pS))$.