Homework 2

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Friday 4th October 2024. Please let me know if any of the problems are unclear or have typos.

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(2.1) Suppose $S \subseteq \mathbb{R}^3$ is a regular surface and $M \subseteq S$. Show that M is a regular surface if and only if M is an open subset of S.

Recall, M is an open subset of S if there exists some open subset $V \subseteq \mathbb{R}^3$ such that $M = S \cap V$.

(2.2) Given a regular surface $S \subseteq \mathbb{R}^3$, define

$$\text{Diff}(S) := \{f : S \to S : f \text{ is a smooth diffeomorphism}\}.$$

a) Show that $\text{Diff}(S)$ forms a group where the binary operation is given by the composition of functions. We call $\text{Diff}(S)$ the diffeomorphism group of S.

b) Let $A : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map

$$A(x, y, z) = (-x, -y, -z), \quad \forall (x, y, z) \in \mathbb{S}^2.$$

Show that $A \in \text{Diff}(\mathbb{S}^2)$, and that A has order 2 in the group.

c) For each $n \in \mathbb{N}$, find a diffeomorphism $f \in \text{Diff}(\mathbb{S}^2)$ with order n in the group.

d) Find a diffeomorphism $f \in \text{Diff}(\mathbb{S}^2)$ with infinite order.

Hint: $O(3) \subseteq \text{Diff}(\mathbb{S}^2)$.

(2.3) Let $S_1, S_2, S_3 \subseteq \mathbb{R}^3$ be regular surfaces, and let $f : S_1 \to S_2$, and $g : S_2 \to S_3$ be smooth functions.

Show that the composition $g \circ f : S_1 \to S_3$ is smooth, and that

$$d(g \circ f)(p) = dg(f(p)) \circ df(p),$$

for any $p \in S_1$.

(2.4) Suppose $S \subseteq \mathbb{R}^3$ is a connected regular surface and $f : S \to \mathbb{R}$ is a smooth function. Show that f is a constant function if and only if $df(p) = 0$ as a linear map, for every $p \in S$.

Hint: as $S \subseteq \mathbb{R}^3$ is a connected regular surface, you may assume that, for any two points $x, y \in S$, there exists $\epsilon > 0$ a smooth regular path $\gamma : (-\epsilon, 1 + \epsilon) \to S$, such that $\gamma(0) = x$ and $\gamma(1) = y$.