Homework 1

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Friday 20th September 2024. Please let me know if any of the problems are unclear or have typos.

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(1.1) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the curve

$$\gamma(t) = (t, \sinh t, \cosh t).$$

a) Calculate the arc-length $L\left(\gamma|_{[0, \frac{1}{2}\log(2)]}\right)$.

b) Calculate its curvature κ.

c) Calculate its torsion τ.

(1.2) Let $u : I \to \mathbb{R}$ be a smooth function, and consider the smooth curve $\gamma_u : I \to \mathbb{R}^2$ defined by

$$\gamma_u(t) = (t, u(t)), \quad \forall t \in I.$$

a) Show that γ_u is a regular curve and describe its trace in terms of the function u.

b) Calculate the curvature of γ_u in terms of the derivatives of u.

c) Choosing $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $u(t) := -\log(\cos(t))$, sketch the trace of γ_u. What is the supremum of the curvature for this example?

(1.3) We take the unit sphere to be the regular surface

$$\mathbb{S}^2 := \{v \in \mathbb{R}^3 : \|v\| = 1\}.$$

Suppose $\gamma : \mathbb{R} \to \mathbb{S}^2$ is a smooth regular curve lying within the unit sphere. Show that the curvature of γ is non-zero everywhere.

Hint: Taylor's theorem

(1.4) Suppose $\gamma : \mathbb{R} \to \mathbb{S}^2$ is a smooth regular curve parameterised by arc-length lying within the unit sphere. By the previous question, its curvature is everywhere non-zero, and so expressing $\gamma(s)$ with respect to the Frenet frame, we find the smooth functions $a, b, c : I \to \mathbb{R}$ such that

$$\gamma(s) = a(s)T(s) + b(s)N(s) + c(s)B(s), \quad \forall s \in I.$$

a) Show that $a \equiv 0$.

b) By differentiating and using the Frenet formulas, show that $b = -\kappa^{-1}$.

c) In the case where the torsion is non-zero, find a formula for c in terms of curvature and torsion.

d) Show that κ and τ solve the differential equation

$$(\kappa')^2 = \kappa^2(\kappa^2 - 1)\tau^2.$$