MATH 2058 - HW 1 - Solutions

Comments.

i. Do NOT skip writing quantifies (for all, for some, there exists, etc) when writing statements. Otherwise, it would be hard for both me and (more importantly) yourself to read the answers.

ii. Inf, Sup are operations for neither functions nor numbers, but sets. Beware of how you take infimums/supremums. In general, make it clear to yourself the kind of objects, for example, sets, functions or numbers, you are dealing with,

1 (P.44-45 Q8)**.** Let $X \subset \mathbb{R}$ be a non-empty subset. Let $f, g : X \to \mathbb{R}$ be functions of bounded ranges.

a. Show that

i. $\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$

ii. $\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}$

b. Give examples to show that each of the above inequalities can either be strict or equal.

Solution. Since f, g are functions of bounded ranges, by the Axiom of Completeness, all the supremums and infimums in questions are finite and well-defined.

a. i. **(Method 1: By Definition of Supremum)** Let $x \in X$. Then $f(x) \leq \sup\{f(x) : x \in X\}$ and $g(x) \leq \sup\{g(x) : x \in X\}$ as supremums are upper bounds. Therefore, we have

$$f(x) + g(x) \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

for all $x \in X$. In other words, $\sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$ is an upper bound for the set $\{f(x) + g(x) : x \in X\}$. By definition of supremum as the *least* upper bound, we have

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

(Method 2: Using an $\epsilon-$argument) Let $\epsilon > 0$. Then by the $\epsilon-$characterization of supremum, there exists $x_0 \in X$ such that

$$\sup\{f(x) + g(x) : x \in X\} - \epsilon < f(x_0) + g(x_0) \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

As the above inequalities holds for arbitrary $\epsilon > 0$, it follows that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

ii. The structure of arguments is basically the same as that of the case of supremums. We present here only the one using an $\epsilon-$argument.
Let $\epsilon > 0$. Then by the $\epsilon-$characterization of infimum, there exists $x_0 \in X$ such that

$$\inf\{f(x) + g(x) : x \in X\} + \epsilon > f(x_0) + g(x_0)$$

Note that $f(x_0) \geq \inf\{f(x) : x \in X\}$ and $g(x_0) \geq \inf\{g(x) : x \in X\}$ as infimums are lower bounds. Hence, we have

$$\inf\{f(x) + g(x) : x \in X\} + \epsilon > f(x_0) + g(x_0) \geq \inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\}$$

As the above inequalities holds for arbitrary $\epsilon > 0$, it follows that

$$\inf\{f(x) + g(x) : x \in X\} \geq \inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\}$$

b. Take $X = \{0, 1\}$. Define $f : X \to \mathbb{R}$ by $f(0) := 0$ and $f(1) := 1$; define $g : X \to \mathbb{R}$ by $g(0) := 1$ and $g(1) := 0$. It is not hard to see that we have

- $\inf\{f(x) : x \in X\} = 0$, $\sup\{f(x) : x \in X\} = 1$
- $\inf\{g(x) : x \in X\} = 0$, $\sup\{g(x) : x \in X\} = 1$
- $\inf\{f(x) + f(x) : x \in X\} = 0$, $\sup\{f(x) + f(x) : x \in X\} = 2$
- $\inf\{f(x) + g(x) : x \in X\} = 1$, $\sup\{f(x) + g(x) : x \in X\} = 1$

Therefore we have

- Equality: $\sup\{f(x) + f(x) : x \in X\} = \sup\{f(x) : x \in X\} + \sup\{f(x) : x \in X\}$
- Equality: $\inf\{f(x) : x \in X\} + \inf\{f(x) : x \in X\} = \inf\{f(x) + f(x) : x \in X\}$
- Strict Inequality: $\sup\{f(x) + g(x) : x \in X\} < \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$
- Strict Inequality: $\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} > \inf\{f(x) + g(x) : x \in X\}$

2 (P.44-45 Q11)**.** Let $X, Y \subset \mathbb{R}$ be nonempty subsets. Let $h : X \times Y \to \mathbb{R}$ be of bounded range. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$, defined by

$$f(x) := \sup\{h(x,y) : y \in Y\} \qquad\qquad g(y) := \inf\{h(x,y) : x \in X\}$$

for all $x \in X$ and $y \in Y$ respectively. Show that

$$\sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}$$

Solution. First, $\sup\{g(y) : y \in Y\}$ and $\inf\{f(x) : x \in X\}$ exist as the respective sets are clearly bounded. Next we present two ways to proceed.
(Method 1: Using an $\epsilon-$argument). Let $\epsilon > 0$. Then there exists $y_0 \in Y$ and $x_0 \in X$ such that $\sup\{g(y) : y \in Y\} - \epsilon/2 < g(y_0)$ and $f(x_0) < \inf\{f(x) : x \in X\} + \epsilon$ by the $\epsilon-$characterizations of supremums and infimums respectively. Note that by definition of g, f, we have

$$g(y_0) = \inf\{h(x, y_0) : x \in X\} \leq h(x_0, y_0) \leq \sup\{h(x_0, y) : y \in Y\} = f(x_0)$$

It follows that

$$\sup\{g(y) : y \in Y\} - \epsilon/2 < g(y_0) \leq f(x_0) < \inf\{f(x) : x \in X\} + \epsilon/2$$

and so

$$\sup\{g(y) : y \in Y\} < \inf\{f(x) : x \in X\} + \epsilon$$

As ϵ is arbitrary, it follows that $\sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}$.

(Method 2: By Definitions) . We claim that for all $x \in X$ and $y \in Y$, we have $g(y) \leq f(x)$. It is because for all $x \in X$ and $y \in Y$, we have by definitions of infimums and supremums as lower and upper bounds that

$$g(y) = \inf\{h(x,y) : x \in X\} \leq h(x,y) \leq \sup\{h(x,y) : y \in Y\} = f(x)$$

Hence, fixing $x \in X$, we have $g(y) \leq f(x)$ for all $y \in Y$. This implies $f(x)$ is an upper bound for $\{g(y) : y \in Y\}$. Therefore, we have $\sup\{g(y) : y \in Y\} \leq f(x)$ by the definition of supremum as the least upper bound . As x is arbitrary, we now have that $\sup\{g(y) : y \in Y\}$ to be a lower bound for $\{f(x) : x \in X\}$. Therefore, by the definition of infimum as the greatest lower bounds, we have

$$\sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}$$