MATH2040A Week 2 Tutorial Notes

1 Span

The *span* of a set of vectors is the collection of all linear combinations of vectors from this set:

$$\text{Span}(\ S\ ) = \left\{ \sum_{i=1}^{n} c_i v_i \ \middle|\ n \in \mathbb{N},\ c_1, \ldots, c_n \in F,\ v_1, \ldots, v_n \in S \right\}$$

In principle, this is similar to the sum of two sets we talked about last tutorial:

$$U_1 + U_2 = \{\ x + y \mid x \in U_1,\ y \in U_2\ \}$$

In fact, you can easily show that the sum of two subspaces is the span of their union $U_1 + U_2 = \text{Span}(\ U_1 \cup U_2\ )$[1].

To check if a vector v is in the span of a set S, we need to check if it is possible to construct such vector by linearly combining vectors from the set. If the set has only finitely many vector $S = \{v_1, \ldots, v_n\}$, we just need to check if the equation

$$v = c_1 v_1 + \ldots + c_n v_n$$

has *some* solution for the scalar coefficients $c_1, \ldots, c_n$. Typically, this would give you a system of linear equations in the coefficients:

- for vectors in $\mathbb{R}^n$, comparing each entry gives you one equation
- for functions, you can compute their values at a few points and compare if they are equal

You can then apply methods e.g. from MATH1030 to solve these linear equations.

Even if you have shown that some solution exists, sometimes it is still *not* enough. Consider the case where $V = P_2(\mathbb{R})$, the space of real polynomials of degree at most 2, $v = x^2$, and $S = \{1, x\}$. If we consider only their values at $x = 0$ and $x = 1$, we obtain the following system for the coefficients a, b of linear combinations $v = a \cdot 1 + b \cdot x$:

$$\begin{cases} a = 0 \\ a + b = 1 \end{cases}$$

Solving this system gives $a = 0$, $b = 1$, and with this it is natural to conclude that $x^2 = a \cdot 1 + b \cdot x = x$ (as polynomials), which is obviously incorrect. For this reason, after solving for the possible coefficients, you should *always* check if they indeed give the desired linear combination.

(On the other hand, showing that a vector is *not* in the span does not have this issue: it suffices to show that no solution to the linear system exists.)

2 Linear independence

A set S of vectors is *linearly dependent* if you can write the zero vector as a nontrivial linear combination of vectors from this set, and *linearly independent* if you cannot. Note that in the definitions, we only care about *nontrivial* linear combinations, that is, combinations where *some* of the coefficients are not zero. This is a necessary constraint as when $c_1 = \ldots = c_n = 0$, we always have $c_1 v_1 + \ldots + c_n v_n = 0 + \ldots + 0 = 0$ for all vectors $v_1, \ldots, v_n$, and this tells us nothing about the vectors.

[1] Textbook Sec 1.4 Q14, homework 2 question. This also explains why the sum of two subspaces is the smallest subspace that contains both.

To show that a set S of vector is linearly dependent, we just need to find some[2] vectors from the set that gives zero with a nontrivial linear combination, that is, find some vectors $v_1, \ldots, v_n \in S$ and scalars $c_1, \ldots, c_n \in F$ not all zero such that $c_1 v_1 + \ldots + c_n v_n = 0$.

On the other hand, to show that a set S of vectors is linearly independent, you need to show that the *only* way to get zero vector as a linear combination of vectors from the set is the trivial combination, that is, whenever you have $c_1 v_1 + \ldots + c_n v_n = 0$ with $v_1, \ldots, v_n \in S$ and $c_1, \ldots, c_n \in F$, you *must* have $c_1 = \ldots = c_n = 0$.

Just like span, you can find a nontrivial linear combination (or show that no nontrivial linear combination exists) by working on the corresponding linear system with methods from e.g. MATH1030.

3 Direct sum

Last tutorial we talk a bit about sum of two subspaces $U_1 + U_2 = \{ x + y \mid x \in U_1, y \in U_2 \}$ and showed some properties about it. A related concept is *direct sum*: if two subspaces U_1, U_2 of a vector space V satisfies the two conditions:

- $U_1 + U_2 = V$
- $U_1 \cap U_2 = \{0\}$

then we say the sum $U_1 + U_2 = V$ is a direct sum and denote it as $V = U_1 \oplus U_2$.

In a sense, direct sum is the linear independence of the subspaces. In particular, it is easy to see the following result: if $V = U_1 \oplus U_2$, and $u_1 \in U_1$, $u_2 \in U_2$ are nonzero, then u_1, u_2 are linearly independent.

4 Exercises

1. (Textbook Sec. 1.4 Q5(a, b)) Determine whether the given vector is in the span of S (in $\mathbb{R}^3$):
   (a) $(2, -1, 1)$, $S = \{ (1, 0, 2), (-1, 1, 1) \}$
   (b) $(-1, 2, 1)$, $S = \{ (1, 0, 2), (-1, 1, 1) \}$

Solution:

(a) Suppose $a, b \in \mathbb{R}$ such that $(2, -1, 1) = a(1, 0, 2) + b(-1, 1, 1) = (a - b, b, 2a + b)$. This gives the linear system

$$\begin{cases} a - b = 2 \\ b = -1 \\ 2a + b = 1 \end{cases}$$

Solving the system we have the solution $a = 1$, $b = -1$. We can verify that $(2, -1, 1) = (1, 0, 2) - (-1, 1, 1)$, so $(2, -1, 1) \in \text{Span}(\ S\ )$.

Note

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

(b) Suppose $a, b \in \mathbb{R}$ such that $(-1, 2, 1) = a(1, 0, 2) + b(-1, 1, 1) = (a - b, b, 2a + b)$. This gives the linear system

$$\begin{cases} a - b = -1 \\ b = 2 \\ 2a + b = 1 \end{cases}$$

We can see that the linear system has no solution, so that is no scalar a, b such that $(-1, 2, 1) = a(1, 0, 2) + b(-1, 1, 1)$. This implies that $(-1, 2, 1) \notin \text{Span}(\ S\ )$.

[2] Finitely many. See also lecture note 3 p.13.

Note

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3/2 \end{array}\right]$$

2. (Textbook Sec 1.5 Q2 (e, f)) Determine whether the following sets are linearly dependent or linearly independent.
   (e) $\{ (1,-1,2), (1,-2,1), (1,1,4) \}$ in $\mathbb{R}^3$
   (f) $\{ (1,-1,2), (2,0,1), (-1,2,-1) \}$ in $\mathbb{R}^3$

Solution:

(e) Let $a, b, c \in \mathbb{R}$ be such that $0 = a(1,-1,2) + b(1,-2,1) + c(1,1,4) = (a+b+c, -a-2b+c, 2a+b+4c)$. Then we have

$$\begin{cases} a+b+c=0 \\ -a-2b+c=0 \\ 2a+b+4c=0 \end{cases}$$

We can see that the system has nontrivial solutions $a = -3c$, $b = 2c$ for all $c \in \mathbb{R}$. In particular, we have $-3(1,-1,2) + 2(1,-2,1) + (1,1,4) = 0$, so the set is linearly dependent.

Note

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 1 \\ 2 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

(f) Let $a, b, c \in \mathbb{R}$ be such that $0 = a(1,-1,2) + b(2,0,1) + c(-1,2,-1) = (a+2b-c, -a+2c, 2a+b-c)$. Then we have

$$\begin{cases} a+2b-c=0 \\ -a+2c=0 \\ 2a+b-c=0 \end{cases}$$

We can see that the system has unique solution $a = b = c = 0$. This implies that the set is linearly independent.

Note

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Let V be a real vector space, $v, w \in V$ be two linearly independent vectors, and $a, b, c, d \in \mathbb{R}$ be scalars such that $ad - bc \neq 0$. Show that $av + bw, cv + dw$ are still linearly independent.

   (Here we are abusing the notation by saying that v, w are linearly independent to mean that the set $\{ v, w \}$ is linearly independent.)

Solution: Let $x, y \in \mathbb{R}$ be two scalars such that $x(av+bw) + y(cv+dw) = 0$. Then $(xa+yc)v + (xb+yd)w = 0$.
Since v, w are linearly independent, we have the system

$$\begin{cases} ax+cy=0 \\ bx+dy=0 \end{cases}$$

or in matrix form

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $ad - bc \neq 0$, the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is invertible, and so the only solution is $x = y = 0$. By definition of x, y, this implies that $av + bw, cv + dw$ are linearly independent.

Note

If we abuse the notation a bit more, we can phrase the proposition in the following way: if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible and v, w are linearly independent, then so are v', w' where $\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$. Of course, the last equation does not *really* make sense.

4. Let V be a vector space, and U_1, U_2 be two subspaces. Show that $V = U_1 \oplus U_2$ if and only if for every vector $v \in V$ there exists unique $u_1 \in U_1, u_2 \in U_2$ such that $v = u_1 + u_2$.

Solution: Suppose $V = U_1 \oplus U_2$.
Then by definition $V = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$. This implies that for every vector $v \in V$ there exists some $u_1 \in U_1$ and $u_2 \in U_2$ such that $v = u_1 + u_2$. It then remains to show that such u_1, u_2 are unique.
Let $v \in V$, $u_1, u_1' \in U_1$, $u_2, u_2' \in U_2$ such that $v = u_1 + u_2$ and $v = u_1' + u_2'$. Then $0 = v - v = (u_1 + u_2) - (u_1' + u_2') = (u_1 - u_1') + (u_2 - u_2')$, which implies $u_1 - u_1' = u_2' - u_2$.
Since U_1, U_2 are subspaces, we have $u_1 - u_1' \in U_1$ and $u_2' - u_2 \in U_2$, so $u_1 - u_1' \in U_1 \cap U_2 = \{0\}$.
This means that $u_1 = u_1'$ and $u_2 = u_2'$, and so such u_1, u_2 are unique.
Therefore, there exists unique $u_1 \in U_1$, $u_2 \in U_2$ such that $v = u_1 + u_2$.

Suppose now that every vector $v \in V$ there exists unique $u_1 \in U_1, u_2 \in U_2$ such that $v = u_1 + u_2$.
Since for each $v \in V$ there exists $u_1 \in U_1$ and $u_2 \in U_2$ such that $v = u_1 + u_2 \in U_1 + U_2$, we have $V = U_1 + U_2$.
Let $v \in U_1 \cap U_2$. As U_1, U_2 are subspaces, we have $0 \in U_1$ and $0 \in U_2$.
Thus, we have the following decompositions for v: $v = v + 0$ with $v \in U_1$ and $0 \in U_2$, and $v = 0 + v$ with $0 \in U_1$ and $v \in U_2$.
By assumption, such representation is unique, so we must have $v = 0$.
This implies that $U_1 \cap U_2 = \{0\}$.
By definition of direct sum, $V = U_1 \oplus U_2$.

Therefore, $V = U_1 \oplus U_2$ if and only if for every vector $v \in V$ there exists unique $u_1 \in U_1, u_2 \in U_2$ such that $v = u_1 + u_2$