Chapter 5: Three topics:

Topic#10 Eigenvalue & Eigenvector

Topic#11 Diagonalizability

Topic#12 Cayley-Hamilton Theorem

Topic#10
Eigenvalue & eigenvectors

<u>Def.</u> Let $T \in \mathcal{L}(V)$.

$0_V \neq v \in V$ is an eigenvector of T **if**

$$\exists \lambda \in \mathbb{F} \text{ s.t. } T(v) = \lambda v.$$

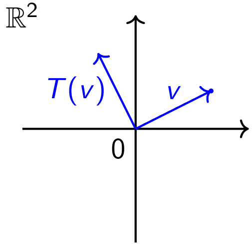
Here, action becomes scalar multiplication.

Here, $\lambda \in \mathbb{F}$ is the **eigenvalue** of $T \in \mathcal{L}(V)$ associated with the (nonzero) eigenvector v.

Examples:

(1) $\exists T \in \mathcal{L}(V)$ which has no eigenvectors.

For instance, $T \in \mathcal{L}(\mathbb{R}^2)$ is a rotation by $\theta = \pi/2$.



Obviously see: for any $0 \neq v \in \mathbb{R}^2$, $T(v)$ can not be a multiple of v.
($\because$ v&$T(v)$ is not colinear)
T has no eigenvectors, hence no eigenvalues.

□

(2) Let $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), f \mapsto T(f) = f'$, where

$$C^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \text{f and its derivatives up to any order are continuous in } \mathbb{R}\}.$$

Note: $T \in \mathcal{L}(C^\infty(\mathbb{R}))$.
Solve: $T(f) = \lambda f, f \neq 0$,
i.e. look for $\lambda \in \mathbb{R}$ and $f \neq 0$ s.t. $f'(t) = \lambda f(t)$.

$\therefore\ f(t) = ce^{\lambda t}(c \neq 0)$.

Then, any $\lambda \in \mathbb{R}$ is an eigenvalue of T, corresponding to the eigenvector $ce^{\lambda t}(c \neq 0)$.

Note: Associated with the eigenvalue $\lambda = 0$, the eigenvector is the nonzero constant function. $\square$

(3) Let $A \in M_{n\times n}$, and $L_A \in \mathcal{L}(\mathbb{F}^n)$. Note: for $0 \neq x \in \mathbb{F}^n$, $\lambda \in \mathbb{F}$

$$L_A(x) = \lambda x \Leftrightarrow Ax = \lambda x.$$

Thus,

Def. $0 \neq x \in \mathbb{F}^n$ is an eigenvector of A if

$$Ax = \lambda x \text{ for some } \lambda \in \mathbb{F}.$$

Here, λ is called the eigenvalue of A corresponding to the eigenvector x.

Def. Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$.

T is **diagonalizable** if

$\exists$ an ordered basis β for V s.t. $[T]_\beta$ is a diagonal matrix.

Thm. Let $T \in \mathcal{L}(V), \dim(V) < \infty$. Then T is diagonalizable **iff** V has an o.b. β in which each basis vector is an eigenvector of T.

Pf. "⇒" Assume: T diagonalizable.

By def., $\exists$ an o.b. β s.t. $[T]_\beta$ is a diagonal matrix.

For $\dim(V) < \infty$, let $\beta = \{v_1, \cdots, v_n\}$, $[T]_\beta = D \overset{def.}{=} \begin{pmatrix} d_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & d_n \end{pmatrix}$.

Then

$$T(v_j) = \sum_{i=1}^{n} D_{ij} v_i = D_{jj} v_j = d_j v_j, j = 1, \cdots, n, \text{ i.e. } T(v_j) = d_j v_j$$

i.e. each vector in β is an e-vector of T. □

"$\Leftarrow$ Let $\beta = \{v_1, \cdots, v_n\}$ be an o.b. for V s.t.

$$T(v_j) = \lambda_j v_j, (1 \leq j \leq n) \text{ for some } \lambda_1, \cdots, \lambda_n \in \mathbb{F}.$$

We see

$$[T]_\beta = ([T(v_1)]_\beta | \cdots | [T(v_n)]_\beta) = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda_n \end{pmatrix}$$

(here, j^{th} column is the β-coord. of $T(v_j)$). □□

Remark. The proof of "$\Leftarrow$" says that

to ensure that T is diagonalizable,
we need to look for a basis of eigenvectors of T,
i.e., to determine the eigenvectors and eigenvalues of T:

$$T(v) = \lambda v, \quad 0 \neq v \in V, \quad \lambda \in \mathbb{F}.$$

e.g. Rotation $T_{\pi/2} \in \mathcal{L}(\mathbb{R}^2)$ has no e-vectors, and thus $T_{\pi/2}$ is NOT diagonalizable.

Observe: Let $T \in \mathcal{L}(V), \dim(V) = n$, β : o.b. for V, then

$$
\begin{aligned}
& T(v) = \lambda v, v \neq 0 \\
& \Leftrightarrow [T(v)]_\beta = \lambda [v]_\beta, [v]_\beta \neq 0 \\
& \Leftrightarrow [T]_\beta [v]_\beta = \lambda [v]_\beta, [v]_\beta \neq 0 \\
& \Leftrightarrow ([T]_\beta - \lambda I_n)[v]_\beta = 0, [v]_\beta \neq 0 \\
& \Leftrightarrow [T]_\beta - \lambda I_n \in M_{n\times n}(\mathbb{F}) \text{ is NOT invertible} \\
& \Leftrightarrow \det([T(v)]_\beta - \lambda I_n) = 0
\end{aligned}
$$

This shows:

Claim: If $T \in \mathcal{L}(V)$ with $\dim(V) < \infty$ and β is an o.b. for V, then λ is an eigenvalue of T **iff**

$$\lambda \text{ is an eigenvalue of } [T]_\beta.$$

e.g. $T_{\pi/2} \in \mathcal{L}(\mathbb{R}^2)$. $T_{\pi/2} = L_A$ with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus

$$0 = det(A - \lambda I_2) = det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

has no solution in $\mathbb{R}$. (Note: $T_{\pi/2} \in \mathcal{L}(\mathbb{R}^2)$ so 'no sol in $\mathbb{R}$')

$\therefore$ A has no eigenvalues
$\therefore$ $T_{\pi/2} = L_A$ has no eigenvalue. $\square$

Def. Let $T \in \mathcal{L}(V)$, $\dim(V) = n$, β : o.b. for V.

$$f_T(t) \overset{def}{=} det([T]_\beta - tI_n)$$

is called the **characteristic polynomial** (c.p.) of T.
i.e. Zeros of $f_T(t)$ give all possible eigenvalues in $\mathbb{F}$ for T.

Remarks:

(1) Note: Matrices $[T]_\beta$ are **similar** for different β's, and similar matrices have the same c.p. Hence, the c.p. $f_T(t) = det([T]_\beta - tI_n)$ is independent of the choice of β, thus we also often write $f_T(t) = det([T]_\beta - tI_n)$.

(2) Let $f_T(t) = det([T]_\beta - tI_n)$. Then

(a) $f_T(t)$ is a poly with $deg = n$ and leading coefficient $(-1)^n$.

(b) $f_T(t)$ has at most n zeros, thus T has at most n e-values. If $\mathbb{F} = \mathbb{C}$, then it has exactly n e-values.

Proof for (1):

$$[T]_\beta = [I_v \circ T \circ I_v]_\beta = [I_v]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}[I_v]_{\beta}^{\beta'} = Q^{-1}[T]_{\beta'}Q$$

$$f_T(t) = det([T]_\beta - tI_n) = det(Q^{-1}[T]_{\beta'}Q - Q^{-1}tI_nQ) = \cdots$$

$$= det(Q^{-1}) \cdot det([T]_{\beta'} - tI_n) \cdot det(Q) = det([T]_{\beta'} - tI_n)$$

A basic fact: (without proof; left for exercises)

Let $T \in \mathcal{L}(V)$. Let $\lambda \in \mathbb{F}$ be an eigenvalue of T. Then $v \in V$ is an eigenvector of T associated with λ **iff**

$$v \neq 0, \text{ and } v \in N(T - \lambda I).$$

Sum: Find e-values & e-vectors of $T \in \mathcal{L}(V)$ with $\dim(V) = n$ & o.b. $\beta = \{v_1, \cdots, v_n\}$ for V.

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & V \\ \Big\downarrow{\scriptstyle [\cdot]_\beta = \Phi_\beta} & & \Big\downarrow{\scriptstyle [\cdot]_\beta = \Phi_\beta} \\ \mathbb{F}^n & \xrightarrow{\quad [T]_\beta \quad} & \mathbb{F}^n \end{array}$$

<u>Recall:</u> $Tv = \lambda v,\ v \neq 0 \Leftrightarrow ([T]_\beta - \lambda I_n)[v]_\beta = 0, [v]_\beta \neq 0.$

1°. Solve $\det([T]_\beta - \lambda I_n) = 0 \Rightarrow$ all eigenvalues λ's of T.

2°. For each λ, find all the λ-e.vectors $x \in \mathbb{F}^n$ by solving

$$([T]_\beta - \lambda I_m)x = 0,$$

then all $v \overset{def}{=} \Phi_\beta^{-1}(x) = \sum_{i=1}^n x_i v_i$ are the λ-e.vectors of T.

e.g. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$

$$f \mapsto T(f), T(f(x)) = f(x) + (1+x)f'(x).$$

Then $T \in \mathcal{L}(P_2(\mathbb{R}))$. Let $\beta = \{1, x, x^2\}$: s.o.b., then

$$A \overset{def}{=} [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$(\because T(1) = 1, T(x) = 1 + 2x, T(x^2) = 2x + 3x^2)$$

1°. Find e-values of T:

$0 = \det([T]_\beta - \lambda I_3) = -(t-1)(t-2)(t-3)$. $\therefore \lambda = 1, 2, 3.$ □

2°. Find e-vectors of T associated with each eigenvalue:

$\lambda_1 = 1$:

$$[T]_\beta - \lambda_1 I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \therefore N([T]_\beta - \lambda_1 I_3) = \text{span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$$

$$\therefore \Phi_\beta^{-1}(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = 1 \cdot 0 + 0 \cdot x + 0 \cdot x^2 = 1$$

is an eigenvector of T associated with $\lambda_1 = 0$. □

$\lambda_2 = 2:$

$$[T]_\beta - \lambda_2 I_3 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \therefore N([T]_\beta - \lambda_2 I_3) = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\}$$

$$\therefore \Phi_\beta^{-1}(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}) = 1 \cdot 0 + 1 \cdot x + 0 \cdot x^2 = 1 + x$$

is an eigenvector of T associated with $\lambda_2 = 2$. □

$\lambda_3 = 3:$

$$[T]_\beta - \lambda_3 I_3 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \therefore N([T]_\beta - \lambda_3 I_3) = \text{span}\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\}$$

$$\therefore \Phi_\beta^{-1}(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}) = 1 \cdot 1 + 2 \cdot x + 1 \cdot x^2 = 1 + 2x + x^2$$

is an eigenvector of T associated with $\lambda_3 = 3$. □

3°. Let

$$\gamma = \{1, 1+x, 1+2x+x^2\},$$

then γ is an o.b. for $P_2(\mathbb{R})$ consisting of only e-vectors of T, and

$$\begin{aligned} T(1) &= 1 \cdot 1, \\ T(1+x) &= 2 \cdot (1+x), \\ T(1+2x+x^2) &= 3 \cdot (1+2x+x^2). \end{aligned}$$

Therefore, T is digonablizable, and

$$[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

□□