Topic#7

Matrix representation of a linear transformation

Def. V: finite-dimensional v.s. over $\mathbb{F}$ with $\dim V = n$
$\beta = \{v_1, v_2, \cdots, v_n\}$ with its order of vectors specified: an ordered basis for V
Let $v \in V$, then $\exists! a_1, \cdots, a_n \in \mathbb{F}$, s.t. $v = \sum_{i=1}^{n} a_i v_i$.
Thus, associated with an ordered basis β for V, we may define

$$[\cdot]_\beta : V \to \mathbb{F}^n$$

such that

$$v \mapsto [v]_\beta \overset{def}{=} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n, \text{ (well-defined)}$$

and $[v]_\beta$ called the **coordinate vector** of v relative to o.b. β or we simply say: $[v]_\beta$ is β-coordinate of v.

Remarks:

1°. $[\cdot]_\beta$ is defined in terms of the o.b. β, so different β's give different $[\cdot]_\beta$'s

e.g.: $V = F^3$: $\beta = \{e_1, e_2, e_3\}$ the standard o.b.

$$\gamma = \{e_2, e_1, e_3\} \text{ o.b.}$$

They are different ordered basis, then $[\cdot]_\beta \neq [\cdot]_\gamma$.

2° $[\cdot]_\beta : V \to \mathbb{F}^n$ with $n = \dim(V)$ is linear, i.e. $[\cdot]_\beta \in \mathcal{L}(V, \mathbb{F}^n)$

(note, to show 'bijection' in the future).

Def. $T \in \mathcal{L}(V, W)$
$\dim(V) = n, \beta = \{v_1, \cdots, v_n\}$: o.b. for V
$\dim(W) = m, \gamma = \{w_1, \cdots, w_m\}$: o.b. for W

$$[T(v_1)]_\gamma = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, [T(v_2)]_\gamma = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, [T(v_n)]_\gamma = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix},$$

$\in \mathbb{F}^m$ are γ-coordinate of $T(v_1) \cdots T(v_n)$, or equivalently

$$T(v_j) = \sum_{i=1}^m a_{ij} w_i, \quad j = 1, 2, \cdots, n$$

where v_j is the j^{th} vector in β and a_{ij} are unique. Then,

$$T \in \mathcal{L}(V, W) \mapsto [T]_\beta^\gamma \overset{def}{=} (a_{ij})_{m \times n} = ([T(v_1)]_\gamma, \cdots, [T(v_n)]_\gamma)$$

is well-defined, and called $[T]_\beta^\gamma$ the **matrix representation** of T in the ordered bases β and γ.
Convention: $[T]_\beta = [T]_\beta^\beta$ if $V = W, \beta = \gamma$

Examples:

(1) $T : \mathbb{R}^2 \to \mathbb{R}^3$
$(a_1, a_2) \mapsto T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$.

$\mathbb{R}^2 : \beta = \{e_1, e_2\}$, s.o.b.
$\mathbb{R}^3 : \gamma = \{e_1, e_2, e_3\}$, s.o.b.

$$T(e_1) = T(1,0) = (1,0,2) = 1e_1 + 0e_2 + 2e_3$$
$$T(e_2) = T(0,1) = (3,0,-4) = 3e_1 + 0e_2 + (-4)e_3$$

$$\therefore [T]_\beta^\gamma = ([T(e_1)]_\gamma, [T(e_2)]_\gamma) = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$

If $\gamma' = \{e_3, e_2, e_1\}$, then

$$[T]_\beta^{\gamma'} = ([T(e_1)]_{\gamma'}, [T(e_2)]_{\gamma'}) = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}.$$

□

(2) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$
$f \in P_3(\mathbb{R}) \mapsto T(f) \in P_2(\mathbb{R}) : T(f(x)) = f'(x)$

$T \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$.

$P_3(\mathbb{R}) : \beta = \{1, x, x^2, x^3\}$ s.o.b.
$P_2(\mathbb{R}) : \beta = \{1, x, x^2\}$ s.o.b.

$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$
$T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$
$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$
$T(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

□

Def. Let $T, U \in \mathcal{L}(V, W)$, and $a \in \mathbb{F}$. We equip $\mathcal{L}(V, W)$ with "$+$" and "$\cdot$" as follows:

$$T + U : V \to W$$

$$x \in V \mapsto (T+U)(x) \overset{def}{=} T(x) + U(x) \in W$$

$$aT : V \to W$$

$$x \in V \mapsto (aT)(x) \overset{def}{=} aT(x)$$

Prop. 1°. $T + U, aT \in \mathcal{L}(V, W)$ (i.e. $\mathcal{L}(V, W)$ is closed under "$+$" and "$\cdot$")

2°. The set $\mathcal{L}(V, W)$ equiped with "$+$" and "$\cdot$" as above is a v.s. over $\mathbb{F}$.

Pf.: Use def. ($+, \cdot$ are well-defined, & (VS1)-(VS8) satisfied). $\square$

Prop. $T, U \in \mathcal{L}(V, W)$.

$$\begin{array}{ccc} V & \xrightarrow[U]{T} & W \\ \Big\downarrow{\scriptstyle [\cdot]_\beta,\dim(V)=n} & & \Big\downarrow{\scriptstyle [\cdot]_\gamma,\dim(W)=m} \\ \mathbb{F}^n & \xrightarrow[B=[U]_\beta^\gamma]{A=[T]_\beta^\gamma} & \mathbb{F}^m \end{array}$$

Then,

$$[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma,$$

$$[aT]_\beta^\gamma = a[T]_\beta^\gamma, \quad a \in \mathbb{F}.$$

Pf.:

$(T+U)(v_j) \overset{1\le j\le n}{=} T(v_j)+U(v_j) = \sum_{i=1}^{m} a_{ij}w_i + \sum_{i=1}^{m} b_{ij}w_i$

$= \sum_{i=1}^{m}(a_{ij}+b_{ij})w_i$

$\therefore ([T+U]_\beta^\gamma)_{ij} = a_{ij}+b_{ij} = ([T]_\beta^\gamma)_{ij} + ([U]_\beta^\gamma)_{ij}$

for $1 \le i \le n, 1 \le j \le m$.

$\therefore [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$ □

Thm. $T \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(W, Z)$, α, β, γ are o.b. for V, W, Z respectively.

$$\begin{array}{ccccc}
V & \xrightarrow{\quad T \quad} & W & \xrightarrow{\quad U \quad} & Z \\
\big\downarrow [\cdot]_\alpha & & \big\downarrow [\cdot]_\beta & & \big\downarrow [\cdot]_\gamma \\
\mathbb{F}^{\dim(V)} & \xrightarrow{\quad [T]_\alpha^\beta \quad} & \mathbb{F}^{\dim(W)} & \xrightarrow{\quad [U]_\beta^\gamma \quad} & \mathbb{F}^{\dim(Z)}
\end{array}$$

Then,

1°. $UT \in \mathcal{L}(V, Z)$, i.e. UT is linear, where

$UT(x) \overset{\forall x \in V}{=} U(T(x))$.

2°.

$$\underbrace{[UT]_\alpha^\gamma}_{\sharp\gamma\times\sharp\alpha} = \underbrace{[U]_\beta^\gamma}_{\sharp\gamma\times\sharp\beta} \; \underbrace{[T]_\alpha^\beta}_{\sharp\beta\times\sharp\alpha} .$$

Proof.

1°. $UT : V \to Z$ is well-defined.

UT is linear. Indeed, $x, y \in V, a \in \mathbb{F}$,

$$\begin{aligned} UT(x+y) &= U(T(x)+T(y)) \\ &= U(T(x)) + U(T(y)) = UT(x) + UT(y), \end{aligned}$$

$$\begin{aligned} UT(ax) &= U(T(ax)) = U(aT(x)) \\ &= aU(T(x)) = aUT(x). \end{aligned}$$

□

2°.

$$\begin{array}{ccccc} V & \xrightarrow{\quad T \quad} & W & \xrightarrow{\quad U \quad} & Z \\ \downarrow [\cdot]_\alpha & & \downarrow [\cdot]_\beta & & \downarrow [\cdot]_\gamma \\ \mathbb{F}^n & \xrightarrow{B_{m\times n} \overset{def}{=} [T]_\alpha^\beta} & \mathbb{F}^m & \xrightarrow{A_{p\times m} \overset{def}{=} [U]_\beta^\gamma} & \mathbb{F}^p \end{array}$$

$$\alpha = \{v_1, \cdots, v_n\} \text{ o.b. for } V, \quad \beta = \{w_1, \cdots, w_m\} \text{ o.b. for } W$$
$$\gamma = \{z_1, \cdots, z_p\} \text{ o.b. for } Z$$

$[U]_\beta^\gamma = A = [a_{ik}]_{p\times m} : U(w_k) = \sum_{i=1}^{p} a_{ik} z_i, 1 \le k \le m,$
$[T]_\alpha^\beta = B = [b_{kj}]_{m\times n} : T(v_j) = \sum_{k=1}^{m} b_{kj} w_k, 1 \le j \le n.$

$$\begin{aligned} \therefore\ UT(v_j) &\overset{j=1,\dots,n}{=} U(\sum_{k=1}^{m} b_{kj} w_k) = \sum_{k=1}^{m} b_{kj} U(w_k) \\ &= \sum_{k=1}^{m} b_{kj} (\sum_{i=1}^{p} a_{ik} z_i) = \sum_{i=1}^{p} (\sum_{k=1}^{m} a_{ik} b_{kj}) z_i \end{aligned}$$

$\therefore ([UT]_\alpha^\gamma)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = (AB)_{ij},\ i = 1, ..., p,\ j = 1, ..., n.$

namely, $[UT]_\alpha^\gamma = AB = [U]_\beta^\gamma [T]_\alpha^\beta$. □□

e.g. $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$, $f \in P_2(\mathbb{R}) \mapsto T(f) \in P_3(\mathbb{R})$, $T(f(x)) = \int_0^x f(t)dt$.
$U : P_3(\mathbb{R}) \to P_2(\mathbb{R})$, $f \in P_3(\mathbb{R}) \mapsto U(f) \in P_3(\mathbb{R})$, $U(f(x)) = f'(x)$.

$$\begin{array}{ccccc} P_2(\mathbb{R}) & \xrightarrow{\quad T \quad} & P_3(\mathbb{R}) & \xrightarrow{\quad U \quad} & P_2(\mathbb{R}) \\ \Big\downarrow [\cdot]_\alpha & & \Big\downarrow [\cdot]_\beta & & \Big\downarrow [\cdot]_\alpha \\ \mathbb{R}^3 & \xrightarrow{\quad [T]_\alpha^\beta \quad} & \mathbb{R}^4 & \xrightarrow{\quad [U]_\beta^\gamma \quad} & \mathbb{R}^3 \end{array}$$

For $T(1) = x$, $T(x) = \frac{1}{2}x^2$, $T(x^2) = \frac{1}{3}x^3$
$u(1) = 0, u(x) = 1, u(x^2) = 2x, u(x^3) = 3x^2$

$$[T]_\alpha^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}_{4\times 3}, \quad [U]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}_{3\times 4}$$

$$[UT]_\alpha = I_{3\times 3} = [U]_\beta^\alpha [T]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

By definition, $UT = I : P_2(\mathbb{R}) \to P_2(\mathbb{R})$

$[UT]_\alpha^\alpha = [I_{P_2(\mathbb{R}^2)}]_\alpha = I_3 = [U]_\beta^\alpha [T]_\alpha^\beta$ □

Remark:

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ \Big\downarrow [\cdot]_\alpha & & \Big\downarrow [\cdot]_\beta \\ \mathbb{F} & \xrightarrow{\quad [T]_\alpha^\beta \quad} & \mathbb{R}^n \end{array}$$

Case $\dim(V) = 1$: $\alpha = \{v\}$ o.b. for V where $v \neq 0$.

For the matrix of T in α & β,

$$[T]_\alpha^\beta = [T(v)]_\beta$$

which is just the coordinate (column) vector of $T(v)$ under β! □

Corollary: Let $T \in \mathcal{L}(V, W)$, where V, W are finite-dimensional with the o.b. β & γ, respectively. Then,

$$\forall v \in V,\ [T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta.$$

$$\begin{array}{ccc} v \in V & \xrightarrow{\quad T \quad} & T(v) \in W \\ \Big\downarrow [\cdot]_\beta & & \Big\downarrow [\cdot]_\gamma \\ [v]_\beta \in \mathbb{F}^m & \xrightarrow{\quad [T]_\beta^\gamma \quad} & [T(v)]_\gamma \in \mathbb{F}^p \end{array}$$

$$m = \dim(V), p = \dim(W)$$

Proof. Take $v \in V$ (fix it!). If $v = 0 \in V$, it is true since $T(v) = T(0_v) = 0_W \Rightarrow [0_W]_\gamma = 0, [0_v]_\beta = 0 \Rightarrow [T]_\beta^\gamma 0 = 0$.

Now let $v \in V$ with $v \neq 0$ Consider

$$
\begin{array}{ccccc}
\mathbb{F} & \xrightarrow{\quad f \quad} & V & \xrightarrow{\quad T \quad} & W \\
\big\downarrow [\cdot]_\alpha & & \big\downarrow [\cdot]_\beta & & \big\downarrow [\cdot]_\gamma \\
\mathbb{F} & \xrightarrow{\quad [f]_\alpha^\beta \quad} & \mathbb{F}^m & \xrightarrow{\quad [T]_\beta^\gamma \quad} & \mathbb{F}^p
\end{array}
$$

Here, $\alpha = \{1\}$ is a basis for $\mathbb{F}$, and

$$f(a) \stackrel{def}{=} av \in V, \forall a \in \mathbb{F}.$$

By Thm, $[Tf]_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta$. Here

$$[Tf]_\alpha^\gamma = [T(f(1))]_\gamma = [T(v)]_\gamma, \quad [f]_\alpha^\beta = [f(1)]_\beta = [v]_\beta.$$

Therefore, $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$. □

Note: for any $v \in V$, $[T]_\beta^\gamma$ can send β-coordinate of $v \in V$ to γ-coordinate of $T(v) \in W$.

Another direct proof: Assume $\dim(V) = n$, and let

$$\beta = \{v_1, v_2, \cdots, v_n\}$$

be an ordered basis for V. Let $v \in V$, then $\exists!\, a_1, \cdots, a_n \in \mathbb{F}$ such that $v = \sum_{j=1}^n a_j v_j$ and hence $T(v) = \sum_{j=1}^n a_j T(v_j)$. Taking the γ-coordinate on both sides,

$$[T(v)]_\gamma = [\sum_{j=1}^n a_j T(v_j)]_\gamma = \sum_{j=1}^n a_j [T(v_j)]_\gamma.$$

Rewrite it as

$$[T(v)]_\gamma = ([T(v_1)]_\gamma, \cdots, [T(v_n)]_\gamma) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

that's

$$[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta. \quad \square$$