

Plan for Chapter 2: (Five topics in total)

Topic #5 Linear transformations

Topic #6 Null space, range, and dimension theorem

Topic #7 Matrix representation of a linear transformation

Topic #8 Invertibility and isomorphism

Topic #9 Change of coordinates

Topic#5

Linear transformations

Def.: Let V, W be v.s. over $\mathbb{F}$. A function $T : V \rightarrow W$ is a **linear transformation** from V to W if $\forall x, y \in V, \forall a \in \mathbb{F}$,

- (1) $T(x + y) = T(x) + T(y)$, and
- (2) $T(ax) = aT(x)$.

We simply say that $T : V \rightarrow W$ is linear.

Notation:

$\mathcal{L}(V, W) \stackrel{\text{def}}{=} \text{set of all linear transformations from } V \text{ to } W.$

$\mathcal{L}(V) \stackrel{\text{def}}{=} \mathcal{L}(V, V)$ in case when $W = V$

Quick Consequences:

(1) If $T : V \rightarrow W$ is linear, then $T(0_V) = 0_W$.

$$\text{Pf.: } T(0_V) = T(0 \cdot 0_V) = 0T(0_V) = 0_W.$$

(2) $T : V \rightarrow W$ is linear **IFF**

$$T(ax + by) = aT(x) + bT(y), \forall x, y \in V, \forall a, b \in \mathbb{F},$$

IFF

$$T\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i T(x_i), \\ \forall x_1, \dots, x_n \in V, \forall a_1, \dots, a_n \in \mathbb{F}.$$

(T preserves the linear combination)

A linear transformation over a finite-dimensional v.s. is uniquely determined by its action on a basis.

Thm.: Let V, W be vector space over $\mathbb{F}$. Assume that V is finite-dimensional with a basis $\{v_1, \dots, v_n\}$. Then, $\forall w_1, \dots, w_n \in W, \exists!$ linear transformation $T : V \rightarrow W$ s.t. $T(v_i) = w_i, i = 1, \dots, n.$

Proof. (Existence)

Let $v \in V = \text{span}(\beta) = \text{span}(\{v_1, \dots, v_n\})$. Then, $\exists! a_1, \dots, a_n \in \mathbb{F}$
s.t. $v = \sum_{i=1}^n a_i v_i$.

Define

$$T(v) \stackrel{\text{def}}{=} \sum_{i=1}^n a_i w_i.$$

Then, $T : V \rightarrow W$ is well-defined.

Claim: $T : V \rightarrow W$ is linear. Proof: to show $\forall u, v \in V, \forall a \in \mathbb{F}$,

(1) $T(u + v) = T(u) + T(v)$, and (2) $T(au) = aT(u)$.

Let

$$u = \sum_{i=1}^n b_i v_i \in V, \quad v = \sum_{i=1}^n c_i v_i \in V,$$

then

$$T(u) = \sum_{i=1}^n b_i w_i, \quad T(v) = \sum_{i=1}^n c_i w_i.$$

Proof of (1): Note

$$u + v = \sum_{i=1}^n b_i v_i + \sum_{i=1}^n c_i v_i = \sum_{i=1}^n (b_i + c_i) v_i$$

By def of T ,

$$\begin{aligned} T(u + v) &= \sum_{i=1}^n (b_i + c_i) w_i \\ &= \sum_{i=1}^n b_i w_i + \sum_{i=1}^n c_i w_i \quad (\text{both} \in W) \\ &= T(u) + T(v). \end{aligned}$$

Proof of (2): Note $au = \sum_{i=1}^n (ab_i) v_i \in V$. Then, by def of T ,

$$T(au) = T\left(\sum_{i=1}^n ab_i v_i\right) = \sum_{i=1}^n (ab_i) w_i = a \sum_{i=1}^n b_i w_i = aT(u).$$



(Uniqueness)

Assume that $\tilde{T} : V \rightarrow W$ is also linear such that

$$\tilde{T}(v_i) = w_i \quad (i = 1, \dots, n),$$

to show: $\tilde{T} = T$, i.e. $\tilde{T}(v) = T(v)$, $\forall v \in V$.

Take $v \in V = \text{span}(\beta)$, $v = \sum_{i=1}^n a_i v_i \in V$. Then

$$\begin{aligned} \tilde{T}(v) &= \tilde{T}\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i \tilde{T}(v_i) \quad (\tilde{T} \text{ is linear}) \\ &= \sum_{i=1}^n a_i w_i \quad (\tilde{T}(v_i) = w_i, i = 1, \dots, n) \\ &\stackrel{\text{existence proof}}{=} T(v) \quad (\text{def. of } T). \end{aligned}$$



Examples:

$$(1) A_{m \times n} \in M_{m \times n}(\mathbb{F}).$$

$\mathbb{F}^n, \mathbb{F}^m$: understood as v.s. of column vectors

Then the function

$$\forall x \in \mathbb{F}^n \mapsto L_A(x) \stackrel{\text{def}}{=} Ax \in \mathbb{F}^m$$

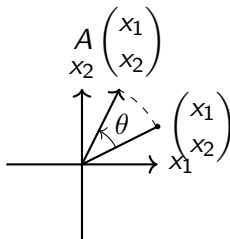
$$\text{Then } (Ax)_i = \sum_{j=1}^n a_{ij}x_j, 1 \leq i \leq m$$

Verify $L_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.

L_A is called a **left-multiplication transformation**.

For instance

- $A \stackrel{\text{def}}{=} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$



$$\begin{aligned} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 &\mapsto L_A(x) = Ax \\ &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{pmatrix} \in \mathbb{R}^2 \end{aligned}$$

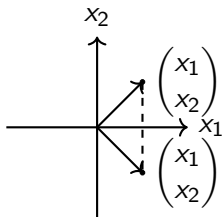
L_A is also called a **rotation** by θ in the anti-clockwise direction. □

- $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mapsto L_A(X) =$$

$$AX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

L_A is also called the **reflection**
about the x_1 -axis. #

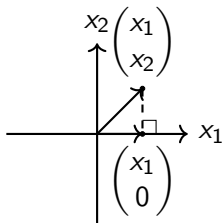


- $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mapsto L_A(X) =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

L_A is also called the **projection**
on the x_1 -axis. □



(2) For any v.s. V and W over $\mathbb{F}$.

$T_0 : V \rightarrow W, \forall x \in V \mapsto T_0(x) \stackrel{\text{def}}{=} 0_W$: the zero transformation.

$I_V : V \rightarrow V, \forall x \in V \mapsto I_V(x) \stackrel{\text{def}}{=} x$: the identity transformation.

(Exercise: $T_0 \in \mathcal{L}(V, W), I_V \in \mathcal{L}(V)$)

(3):

(a) $T: M_{m \times n}(\mathbb{F}) \mapsto M_{m \times n}(\mathbb{F}) \quad A \mapsto T(A) \stackrel{\text{def}}{=} A^t$

defines a linear transformation from $M_{m \times n}(\mathbb{F})$ to $M_{m \times n}(\mathbb{F})$.

(b) $T: f \in P_n(\mathbb{R}) \mapsto T(f) \in P_{n-1}(\mathbb{R})$ given by

$$T(f)(x) = f'(x), \forall x \in \mathbb{R},$$

defines a linear transformation from $P_n(\mathbb{R})$ to $P_{n-1}(\mathbb{R})$.

(c)

$T: C(\mathbb{R}) \mapsto \mathbb{R}, f \mapsto T(f) \stackrel{\text{def}}{=} \int_a^b f(t)dt \in \mathbb{R}, (-\infty < a < b < \infty)$

defines a linear transformation from $C(\mathbb{R})$ to $\mathbb{R}$.