Topic#3
Span & linear (in-)dependence

$(V, +, \cdot)$: v.s. over $\mathbb{F}$

Def.

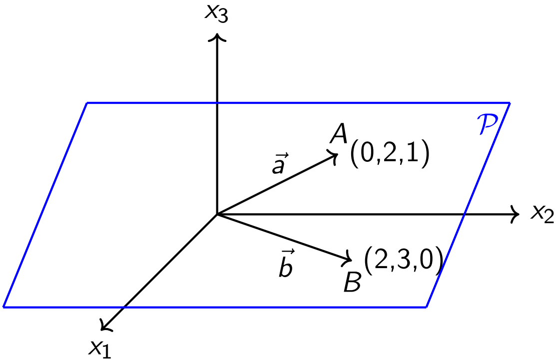
- $$\sum_{i=1}^{m} a_i v_i = a_1 v_1 + \cdots + a_m v_m \in V$$

  is called a **linear combination** of $v_1, \cdots, v_m \in V$ with combination coefficients $a_1, \cdots, a_m \in \mathbb{F}$.

- Let $\emptyset \neq S \subset V$.

  span$(S) \overset{def}{=}$ a set of all possible linear combinations of vectors in S

  $$= \{\sum_{i=1}^{m} a_i v_i : \text{each } a_i \in \mathbb{F}, \text{each } v_i \in S, 1 \leq i \leq m, m = 1, 2, \cdots\}$$



$\mathcal{P}$ consists of all vectors of the form

$$s\vec{a} + t\vec{b} \text{ (linear combination)}, \quad s, t \in \mathbb{R}$$

$$\mathcal{P} = span\{\vec{a}, \vec{b}\} \text{ (span of } \{\vec{a}, \vec{b}\})$$

Note:

(1) Any linear combination contains only finite many terms.

(2) Even if S is infinite, we can still define $\text{span}(S)$ well.

(3) Convention: if $S = \emptyset$, then $\text{span}(\emptyset) = \{0\}$.

An example: $f, f_1, f_2 \in \mathbb{P}_3(\mathbb{R})$: How to find $a, b \in \mathbb{R}$ s.t.
$f = af_1 + bf_2$?
(generally, $v \in V = \sum_{i=1}^{m} a_i v_i$, how to determine a_i?)

Claim: it is equivalent to solve a linear system $Ax = b$
$f = 2x^3 - 2x^2 + 12x - 6$, $f_1 = x^3 - 2x^2 - 5x - 3$,
$f_2 = 3x^3 - 5x^2 - 4x - 9$.

Plug the three equations to $f = af_1 + bf_2$.

Get: $2 = a + 3b$, $-2 = -2a - 5b$, $12 = -5a - 4b$, $-6 = -3a - 9b$.

$\Rightarrow \exists! a = -4 \in \mathbb{R}, b = 2 \in \mathbb{R}$ $\square$

Prop. $(V, +, \cdot)$: v.s. over $\mathbb{F}$. $S \subset V$. Then

(1). span(S) is a subspace of V, and

(2). span(S) is the smallest subspace of V containing S in the sense that if W is a subspace with $W \supset S$, then

$$W \supset \text{span}(S).$$

namely, any subspace containing S must contain span(S).

<u>Proof.</u> $S = \emptyset$, by convention, $\text{span}(\emptyset) = \{0\}$ is a subspace of V
Of course is a smallest subspace of V containing $\emptyset$.

Assume $V \supset S \neq \emptyset$.
(1) to show $\text{span}(S)$ is a subspace of V. Indeed,

(a) $\text{span}(S) \subset V$.
In fact, take $v \in \text{span}(S)$.
By def of span, $v = \sum_{i=1}^{m} a_i v_i$ with $v_i \in S$, $a_i \in \mathbb{F}$.
$\because S \subset V \therefore$ all $v_i \in V$.
$\therefore$ Since V is a v.s., $v = \sum_{i=1}^{m} a_i v_i \in V$. $\square$

(b) $0 \in \text{span}(S)$
Indeed, $S \neq \emptyset$, $\exists v \in S$. $0 = 0v \in \text{span}(S)$
since LHS is zero of V, 0 at RHS is zero scalar of $\mathbb{F}$ and $v \in S$. $\square$

(c) Let $u, v \in \text{span}(S)$, to show $u + v \in$span(S). Indeed, $\because u, v \in$span(S)

$$u = \sum_{i=1}^{m} a_i u_i, \quad v = \sum_{i=1}^{n} b_i v_i \quad \text{with} a_i, b_i \in \mathbb{F}, u_i, v_i \in S,$$

$$\text{then } u + v = (a_1 u_1 + \cdots + a_m v_m) + (b_1 v_1 + \cdots + b_n v_n)$$

$$\text{is still a linear combination of vectors}$$
$$u_1, \cdots, u_m, v_1, \cdots, v_n \in \text{span}(S) \text{ and } a_1, \cdots, a_m, b_1, \cdots, b_n \in \mathbb{F}.$$

$\therefore u + v \in$span(S).

(d) Let $a \in \mathbb{F}$, $u \in \text{span}(S)$. Let $u = \sum_{i=1}^{m} a_i u_i$ for $a_i \in \mathbb{F}, u_i \in S$. Then for $a \in \mathbb{F}$,

$$au = a(\sum_{i=1}^{m} a_i u_i) = \sum_{i=1}^{m} (aa_i) u_i \in \text{span}(S) \quad \text{since } aa_i \in \mathbb{F}, u_i \in S$$

It is a linear combination.

$\therefore$ span(S) is a subspace of V. □

(2) Assume W is a subspace of V and $W \supset S$,
to show: $W \supset \text{span}(S)$.

Take $v \in \text{span}(S)$. Then,

$$v = \sum_{i=1}^{m} a_i v_i, \quad a_i \in \mathbb{F}, v_i \in S.$$

Since $S \subset W$ all $v_i \in W$

$\because$ W is a subspace of V i.e. $W \subset V$ is a v.s., each $v_i \in W$.

$$\therefore v = \sum_{i=1}^{m} a_i v_i \in W.$$

$\therefore \text{span}(S) \subset W$. $\square$

<u>Def.</u> $(V, +, \cdot)$: v.s. over $\mathbb{F}$. $S \subset V$. We say that S **spans** V if

$$V = \text{span}(S) \text{ for some } S \subset V$$

e.g. * $\mathbb{F}^n = \text{span}\{e_1, \cdots, e_n\}$,

where $e_i = (0, \cdots, 0, \underset{i^{th}}{\underset{\uparrow}{1}}, 0, \cdots)$

* $P_n(\mathbb{F}) = \text{span}(\{1, x, x^2, \cdots, x^n\})$,

$P(\mathbb{F}) = \text{span}(\{1, x, x^2, \underbrace{\cdots}_{\text{infinite}}\})$,

* $M_{m \times n}(\mathbb{F}) = \text{span}(\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\})$

where E_{ij} is the matrix with all zero entries except 1 at i^{th} row and j^{th} column.

Basic question: V: v.s. over $\mathbb{F}$:

(1). Does V have a finite spanning set?

(2). If so, can one find a finite spanning set with the min size?
(linearly (in)dependence)

$(V, +, \cdot)$: v.s. over $\mathbb{F}$

Def. $S \subset V$ is **linearly dependent** if $\exists$ distinct $v_1, \cdots, v_m \in S$ and $a_1, \cdots, a_m \in \mathbb{F}$ (not all zero), s.t.

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Otherwise $S \subset V$ is **linearly independent**.

Remarks:

(1) $\emptyset \subset V$ is l. indep.; Any l. dep. subset of V must be non-empty.

(2) If $0 \in S \subset V$, then S is l. dep. ($\because 1 \cdot 0 = 0$)

(3) $S = \{v\}$ is l. indep. $\Leftrightarrow v \neq 0$.

More observations. Let $S_1 \subset S_2 \subset V$, then

(a) $\text{span}(S_1) \subset \text{span}(S_2)$

(b) if S_1 l. dep. then S_2 l. dep.

(S_1 l.dep. $\overset{def}{\Rightarrow} \exists$ distinct $v_1, \cdots, v_m \in S_1 \subset S_2$
and $a_1, \cdots, a_m \in \mathbb{F}$ (Not all zero) s.t. $a_1 v_1 + \cdots + a_m v_m = 0$).

(c) If $V = \text{span}(S_1)$ then $V = \text{span}(S_2)$

Lemma. Let $S \subset V$.

(1).S is l. indep **iff** any finite subset of S is l. indep.

Proof: "$\Rightarrow$" Otherwise, $\cdots$

"$\Leftarrow$" Otherwise, S is l.dep., then by def., $\exists$ distinct $v_1, \cdots, v_m \in S$ and $a_1, \cdots, a_m \in \mathbb{F}$ (not all zero) s.t. $a_1 v_1 + \cdots + a_m v_m = 0$).
Def $S_1 \stackrel{def}{=} \{v_1, \cdots, v_m\} \subset S$ contradiction with S_1 is l.indep. $\square$

(2).Let $S = \{v_1, v_2, \cdots, v_n\}$ be a finite subset of V. Then, the following three are equivalent:
(a).S is l. indep.

(b).If $\sum_{i=1}^{n} a_i v_i = 0$ ($a_i \in F$) then $a_1 = \cdots = a_n = 0$.

(c).If $v = \sum_{i=1}^{n} a_i v_i \in$span$(S)$ ($a_i \in F$) then $a_1, \cdots, a_n$ are unique.

Proof:

(a)$\Leftrightarrow$(b):

"$\Rightarrow$" Note: when S is l.indep., $v_1, \cdots, v_n$ are distinct. Let $\sum_{i=1}^{m} a_i v_i = 0 (a_i \in \mathbb{F})$, to show: $a_1 = \cdots = a_n = 0$.
Indeed, otherwise, not all a_i are zero.
$\therefore S = \{v_1, \cdots, v_n\}$ is l.dep. by def.. Contradiction!

"$\Leftarrow$" Otherwise, S is linearly dependent.

(b)$\Leftrightarrow$(c):

"$\Leftarrow$" Let $\sum_{i=1}^{n} a_i v_i = 0$. Note: $\sum_{i=1}^{n} 0 v_i = 0 = \sum_{i=1}^{n} a_i v_i$.
By (c), i.e. by uniqueness of a_i, $a_1 = \cdots = a_n = 0$.

"$\Rightarrow$" Let $v = \sum_{i=1}^{n} a_i v_i \in \text{span}(S)$, to show: $a_1, \cdots, a_n$ are unique.
Indeed, let $v = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i, (b_i \in \mathbb{F})$
$\therefore \sum_{i=1}^{n} (a_i - b_i) v_i = 0$.
By (b), $a_i - b_i = 0$ for each i, i.e. $a_i = b_i, 1 \leq i \leq n$. $\square$.

Thinking:
(1). span($\{v \neq 0\}$)=V, otherwise $\not\subset V, \cdots$
(2).$V = \text{span}(V)$, kick away some vectors of V without changing span.

Prop. $(V, +, \cdot)$: v.s. over $\mathbb{F}$. $S \subset V$ is l. dep. Then

$$\exists v \in S \text{ s.t. } \text{span}(S) = \text{span}(S \setminus \{v\}).$$

i.e. if S is l.dep., then one can remove at least one vector in S without changing its span.

Proof. S l. dep $\Rightarrow \exists$ distinct $v_1, \cdots, v_m \in S$ & $a_1, \cdots, a_m \in \mathbb{F}$ (not all zero) s.t. $a_1 v_1 + \cdots + a_m v_m = 0$. For instance $a_1 \neq 0$, then

$$v_1 = -\frac{a_2}{a_1} v_2 - \cdots - \frac{a_m}{a_1} v_m \text{ with } -\frac{a_2}{a_1}, \cdots, -\frac{a_m}{a_1} \in \mathbb{F}$$

Then choose $v = v_1$, then

$$\text{span}(S) \subset \text{span}(S \setminus \{v\})$$

Because: "$\supset$": $S \setminus \{v\} \subset S (\because v \in S)$
"$\subset$": Let $u \in \text{span}(S)$, then
$u = b_1 u_1 + \cdots + b_n u_n, b_1, \cdots, b_n \in \mathbb{F}, u_1, \cdots, u_n \in S.$

In case, some of u_i is $v = v_1$, then one can replace such u_i by $u_i \in \text{span}(\{v_2, \cdots, v_n\})$ where $\{v_2, \cdots, v_n\} \subset S \setminus \{v\}$.

Then, $u \in \text{span}(S \setminus \{v\})$. □

Prop. $(V, +, \cdot)$: v.s. over $\mathbb{F}$. $S \subset V$ is l. indep., $v \in V \setminus S$. (i.e. $v \notin S, v \in V$). Then $S \cup \{v\}$ is l. dep. **iff** $v \in \text{span}(S)$.

Proof. "$\Rightarrow$" Assume: $S \cup \{v\}$ is l. dep., to show: $v \in \text{span}(S)$.
Indeed, by def, $\exists$ distinct $u_1, \cdots, u_m \in S \cup \{v\}$ and $a_1, \cdots, a_m \in \mathbb{F}$ (not all zero)

$$\text{s.t.} \quad a_1 u_1 + \cdots + a_m u_m = 0. \qquad (*)$$

Claim: At lease one of u_i should be v.
(otherwise, no u_i is v, it means that all $u_1, \cdots, u_m \neq v$, then $u_1, \cdots, u_m$ are from S. Note: S is l.indep. then (*) is a contradiction)

□

For instance, assume $u_1 = v$ & $a_1 \neq 0$

$$\therefore v = u_1 = (-\frac{a_2}{a_1})\underset{\substack{\uparrow \\ \in S}}{u_2} + \cdots + (-\frac{a_m}{a_1})\underset{\substack{\uparrow \\ \in S}}{u_m} \in \text{span}(S)$$

(Note: $\because u_2, \cdots, u_m \in S \cup \{v\}$ are distinct with $u_1 = v$
$\therefore u_2, \cdots, u_m \in S$.)

"$\Leftarrow$" Let $v \in \text{span}(S)$ to show: $S \cup \{v\}$ is linearly dep.
Indeed, $\because v \in \text{span}(S)$
$\therefore v = \sum_{i=1}^{m} a_i v_i, v_i \in S, a_i \in \mathbb{F}$. (w.l.g., all v_i distinct)

Namely, $a_1 v_1 + \cdots + a_m v_m + (-1)v = 0$
(i) $v_1, \cdots, v_m, v \in S \cup \{v\}$ distinct ($\because v \notin S$).
(ii) $a_1, \cdots, a_m, -1(\neq 0)$: not all zero

By def, $S \cup \{v\}$ is l. dep. □