Topic#1
Vector Space

Linear algebra I | **II**

Linear algebra I		II
Euclidean space $\mathbb{R}^n$, $n = 1, 2, \cdots$	$\leftrightarrow$	Vector space V (abstract)
Matrix $M_{m\times n} \underset{\in\mathbb{R}^n}{x} = \underset{\in\mathbb{R}^m}{y}$ $M_{m\times n} : \mathbb{R}^n \to \mathbb{R}^m$	$\leftrightarrow$	Linear transformaton $T : V \to W$ $\dim V = n, dim W = m$

Notion:

$\mathbb{R}$ = set of all real numbers

$\mathbb{C}$ = set of all complex numbers $= \{z = a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$

Through the course, assume: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$
(generally $\mathbb{F}$ denotes a field)

Def. Vector space

A vector space V over $\mathbb{F}$ is a set over which **addition** and **scalar multiplication**

$$\begin{aligned} + : \quad & V \times V \to V \\ & (x, y) \mapsto x + y \\[1em] \cdot : \quad & \mathbb{F} \times V \to V \\ & (a, x) \mapsto ax \end{aligned}$$

are well-defined and the following conditions $(VS1) - (VS8)$ hold:

(VS1) commutativity of $+$: $x + y = y + x$, $\forall x, y \in V$

(VS2) associativity of $+$: $(x + y) + z = x + (y + z)$, $\forall x, y, z \in V$

(VS3) zero for $+$: $\exists 0 \in V$ s.t. $x + 0 = x$, $\forall x \in V$

(VS4) inverse for $+$: $\forall x \in V$, $\exists y \in V$ s.t. $x + y = 0$

(VS5) unit for $\cdot$: $1x = x$, $\forall x \in V$

(VS6) associativity of $\cdot$: $(ab)x = a(bx)$, $\forall a, b \in \mathbb{F}$, $\forall x \in V$

(VS7) distributive law #1: $a(x + y) = ax + ay$, $\forall a \in \mathbb{F}$, $\forall x, y \in V$

(VS8) distributive law #2: $(a + b)x = ax + bx$, $\forall a, b \in \mathbb{F}$, $\forall x \in V$

$\square$

Note:

V: a vector space ($v.s.$)

$(V, +, \cdot)$: to emphasize "$+$" and "$\cdot$"

V over $\mathbb{F}$: to emphasize $\mathbb{F}$

$\mathbb{F} = \mathbb{R}$: V is a real v.s.

$\mathbb{F} = \mathbb{C}$: V is a complex v.s.

$x \in V$: x is a vector in V

$a \in \mathbb{F}$: a is a scalar in $\mathbb{F}$

Examples:

(1) $V = \mathbb{F}^n$:

$$x \overset{def}{=} (x_1, x_2, \cdots, x_n) \in V$$
$$y \overset{def}{=} (y_1, y_2, \cdots, y_n) \in V$$
$$a \in \mathbb{F}$$

$$x + y \overset{def}{=} (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$
$$ax \overset{def}{=} (ax_1, ax_2, \cdots, ax_n)$$

Verify: $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ with $+$ and $\cdot$ defined as above.

* $+. \cdot$ are well-defined
* (VS1)-(VS8) hold

(2) **Polynomials:**
$0, 1, x, x^2, x^3, \cdots, x^n, \cdots$

Given two polynomials:

$$f(x) = \sum_{i=0}^{n} a_i x^i, \quad g(x) = \sum_{i=0}^{m} b_i x^i,$$

$f(x) = g(x)$ iff $n = m$ & $a_i = b_i$ $(0 \leq i \leq n)$.

$P_n(\mathbb{F}) \overset{def}{=}$ set of all polynomials with coefficients in $\mathbb{F}$ and degree at most n (degree $\leq n$)$=\{\sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{F}\}$

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i \overset{def}{=} \sum_{i=0}^{n} (a_i + b_i) x^i$$

$$a \sum_{i=0}^{n} a_i x^i \overset{def}{=} \sum_{i=0}^{n} (a a_i) x^i$$

$(P_n(\mathbb{F}), +, \cdot)$ is a v.s. over $\mathbb{F}$

$P(\mathbb{F}) \stackrel{def}{=}$ set of all polynomials of any order with coefficients in $\mathbb{F}$

$+, \cdot$: defined as before

$(P(\mathbb{F}), +, \cdot)$ is a vector space over $\mathbb{F}$

(3) **Matrix:**

$M_{m\times n}(\mathbb{F})$ = set of all $m \times n$ matrices with entries in $\mathbb{F}$

$+, \cdot$; defined in the usual way, i.e.

$A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, $a \in \mathbb{F}$

$A + B \stackrel{def}{=} (a_{ij} + b_{ij})_{m\times n}$

$aA \stackrel{def}{=} (aa_{ij})_{m\times n}$

$(M_{m\times n}(\mathbb{F}), +, \cdot)$ is a v.s. over $\mathbb{F}$

Other interesting examples:

(4) $V = \{0\}$: set consisting of zero only

$$0 + 0 = 0, \quad a0 = 0$$

zero v.s. (smallest v.s.)

(5) $\mathbb{F}^\infty \overset{def}{=} \{(x_1, x_2, \cdots) : x_i \in \mathbb{F}, i \geq 1\}$
$+, \cdot$: usual way

(6) $\mathbb{F}^S$=set of all functions on $S(\neq \emptyset)$ with values in $\mathbb{F}$

$$+ : (f + g)(x) \overset{def}{=} f(x) + g(x), \ x \in S$$

$$\cdot : (af)(x) = af(x), \ x \in S.$$

Basic properties of vector spaces:

(1) 0 is unique.

(2) $\forall x \in V$, $\exists y \in V$ s.t. $x + y = 0$, where y is unique.

(3) $x + z = y + z \Rightarrow x = y$ (cancellation law).

(4) $0x = 0$, $\forall x \in V$.

(5) $a0 = 0$, $\forall a \in \mathbb{F}$.

(6) $(-1)x = -x$.

Proof. (1) Let $0' \in V$ be s.t. $x + 0' = x$, $\forall x \in V$. Then

$$\begin{aligned} 0' &= 0' + 0 \text{ (0 is zero)} \\ &\overset{(VS1)}{=} 0 + 0' \text{ ($0'$ is zero)} \\ &= 0. \end{aligned}$$

(2) Let $x \in V$. Assume: $\exists y, y' \in V$, s.t. $x + y = 0 = x + y'$.

$$\begin{aligned} y' &= y' + 0 \\ &= 0 + y' \\ &= (x + y) + y' \\ &= (y + x) + y' \\ &= y + (x + y') \\ &= y + 0 \\ &= y. \end{aligned}$$

Convention: If $x + y = 0$, then we write $y = -x$ as the additive inverse of x.

(3)

$$\begin{aligned} x + z = y + z &\Rightarrow (x + z) + (-z) = (y + z) + (-z) \\ &\Rightarrow x + (z + (-z)) = y + (z + (-z)) \\ &\Rightarrow x + 0 = y + 0 \\ &\Rightarrow x = y. \end{aligned}$$

(4)

$$0 + 0x = 0x = (0 + 0)x = 0x + 0x \Rightarrow 0x = 0.$$

(5)&(6): Exercise. □