THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A
Solution to Homework 6
10 points for each question

Compulsory Part

Sec. 2.4

14 Q: Let

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}.$$

Construct an isomorphism from V to F^3.

Sol: Define $T : V \to \mathsf{F}^3$ by $\forall A \in V$, $T(A) = (A_{11}, A_{12} - A_{11}, A_{22})$. We first show that T is a linear transformation. Indeed, $\forall A, B \in V$ and $\forall a \in \mathsf{F}$,

$$\begin{aligned} T(A+B) &= (A_{11} + B_{11}, (A_{12} + B_{12}) - (A_{11} + B_{11}), A_{22} + B_{22}) \\ &= (A_{11}, A_{12} - A_{11}, A_{22}) + (B_{11}, B_{12} - B_{11}, B_{22}) = T(A) + T(B), \\ T(aA) &= (aA_{11}, aA_{12} - aA_{11}, aA_{22}) = a(A_{11}, A_{12} - A_{11}, A_{22}) = aT(A). \end{aligned}$$

Suppose $A \in V$ and $T(A) = 0$. Then $(A_{11}, A_{12} - A_{11}, A_{22}) = (0, 0, 0)$, implying that $A_{11} = A_{12} = A_{22} = 0$. Also, as $A \in V$, $A_{21} = 0$. Hence, A is the 2×2 zero matrix over F. This shows that $T : V \to \mathsf{F}^3$ is one-to-one.
Also, $T : V \to \mathsf{F}^3$ is onto because $\forall a, b, c \in F$,

$$T \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, (a+b) - a, c) = (a, b, c).$$

Therefore, T is an isomorphism from V to F^3.

15 Q: Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Sol: Write $\beta = \{u_1, ..., u_n\}$, where $u_1, ..., u_n$ are distinct vectors in V.
($\Rightarrow$) Suppose T is an isomorphism. Then $T(u_1), ..., T(u_n)$ are distinct. Suppose $a_1, ..., a_n$ are scalars such that

$$a_1 T(u_1) + \cdots + a_n T(u_n) = \vec{0}.$$

Then

$$a_1 u_1 + \cdots + a_n u_n = T^{-1}(a_1 T(u_1) + \cdots + a_n T(u_n)) = T^{-1}(\vec{0}) = \vec{0}.$$

As β is a basis for V and in particular linearly independent, $a_1 = \cdots = a_n = 0$. Thus, $T(\beta) = \{T(u_1), ..., T(u_n)\}$ is also linearly independent.
Since $\dim W = n$ and the cardinality of $T(\beta)$ is also n, $T(\beta)$ is a basis for V.
(**Alternatively**, suppose $w \in W$. Then $\exists$ scalars $a_1, ..., a_n$ such that $T^{-1}(w) = \sum_{i=1}^n a_i u_i$ and hence $w = T(T^{-1}(w)) = \sum_{i=1}^n a_i T(u_i) \in \operatorname{span} T(\beta)$. Thus, $T(\beta)$ spans W. To conclude, $T(\beta)$ is a basis for W.)

16 Q: Let B be an $n \times n$ invertible matrix. Define $\Phi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Sol: $\forall A, A' \in \mathsf{M}_{n\times n}(F)$ and $\forall a \in F$,

$$\begin{aligned}\Phi(A + A') &= B^{-1}(A + A')B = B^{-1}AB + B^{-1}A'B = \Phi(A) + \Phi(A') \\ \Phi(aA) &= B^{-1}(aA)B = a(B^{-1}AB) = a\Phi(A).\end{aligned}$$

Hence, Φ is a linear transformation.

Method 1: Suppose $A \in \mathsf{M}_{n\times n}(F)$ and $\Phi(A)$ is the $n \times n$ zero matrix O over F. Then $A = B\Phi(A)B^{-1} = BOB^{-1} = O$. Hence, $\Phi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ is one-to-one. $\forall C \in \mathsf{M}_{n\times n}(F)$, $BCB^{-1} \in \mathsf{M}_{n\times n}(F)$ and $\Phi(BCB^{-1}) = B(B^{-1}CB)B^{-1} = C$. Hence, $\Phi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ is also onto. Therefore, Φ is an isomorphism.

Method 2: Suppose $A \in \mathsf{M}_{n\times n}(F)$ and $\Phi(A)$ is the $n \times n$ zero matrix O over F. Then $A = B\Phi(A)B^{-1} = BOB^{-1} = O$. Hence, $\Phi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ is one-to-one. By Theorem 2.5 in Sec. 2.1, Φ is also onto. Therefore, Φ is an isomorphism.

Method 3: $\forall C \in \mathsf{M}_{n\times n}(F)$, $BCB^{-1} \in \mathsf{M}_{n\times n}(F)$ and $\Phi(BCB^{-1}) = B(B^{-1}CB)B^{-1} = C$. Hence, $\Phi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ is onto.
By Theorem 2.5 in Sec. 2.1, Φ is also one-to-one. Therefore, Φ is an isomorphism.

Sec. 2.5

2 Q: For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$.

Sol: (d) Let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then

$$\begin{cases}(2, 1) &= Q_{11}(-4, 3) + Q_{21}(2, -1); \\ (-4, 1) &= Q_{12}(-4, 3) + Q_{22}(2, -1).\end{cases}$$

We rewrite this system into matrix form:

$$\begin{pmatrix}2 & -4 \\ 1 & 1\end{pmatrix} = \begin{pmatrix}-4 & 2 \\ 3 & -1\end{pmatrix} Q.$$

On solving,

$$Q = \begin{pmatrix}-4 & 2 \\ 3 & -1\end{pmatrix}^{-1} \begin{pmatrix}2 & -4 \\ 1 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 1 \\ \frac{3}{2} & 2\end{pmatrix} \begin{pmatrix}2 & -4 \\ 1 & 1\end{pmatrix} = \begin{pmatrix}2 & -1 \\ 5 & -4\end{pmatrix}.$$

3 Q: For each of the following pairs of ordered bases β and β' for $\mathsf{P}_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(f) $\beta = \{2x^2 - x + 1, x^2 + 3x - 2, -x^2 + 2x + 1\}$ and $\beta' = \{9x - 9, x^2 + 21x - 2, 3x^2 + 5x + 2\}$.

Sol: (f) Let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then

$$\begin{cases} 9x - 9 & = Q_{11}(2x^2 - x + 1) + Q_{21}(x^2 + 3x - 2) + Q_{31}(-x^2 + 2x + 1); \\ x^2 + 21x - 2 & = Q_{12}(2x^2 - x + 1) + Q_{22}(x^2 + 3x - 2) + Q_{32}(-x^2 + 2x + 1); \\ 3x^2 + 5x + 2 & = Q_{13}(2x^2 - x + 1) + Q_{23}(x^2 + 3x - 2) + Q_{33}(-x^2 + 2x + 1). \end{cases}$$

We rewrite this system into matrix form:

$$\begin{pmatrix} 0 & 1 & 3 \\ 9 & 21 & 5 \\ -9 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 2 \\ 1 & -2 & 1 \end{pmatrix} Q.$$

On solving,

$$\begin{aligned} Q &= \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 2 \\ 1 & -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 3 \\ 9 & 21 & 5 \\ -9 & -2 & 2 \end{pmatrix} \\ &= \frac{1}{18} \begin{pmatrix} 7 & 1 & 5 \\ 3 & 3 & -3 \\ -1 & 5 & 7 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 9 & 21 & 5 \\ -9 & -2 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}. \end{aligned}$$

4 Q: Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix}.$$

Let β be the standard ordered basis for $\mathbb{R}^2$, and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

to find $[T]_{\beta'}$.

Sol: We first find out the change of coordinate matrix Q that changes β'-coordinates into β-coordinates, which is

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that we have

$$[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}.$$

Now, by Theorem 2.23 in Sec. 2.5,

$$\begin{aligned} [T]_{\beta'} = Q^{-1}[T]_\beta Q &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}. \end{aligned}$$

6 Q: For each matrix A and ordered basis β, find $[\mathsf{L}_A]_\beta$. Also, find an invertible matrix Q such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

(d) $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$ and $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Sol: (d) By the Corollary in page 115 in Sec. 2.5,

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

is in invertible matrix such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$. Note that

$$Q^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}.$$

Hence, we get

$$[\mathsf{L}_A]_\beta = \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}.$$

7 Q: In $\mathbb{R}^2$, let L be the line $y = mx$ where $m \neq 0$. Find an expression for $T(x, y)$ where

(a) T is the reflection of $\mathbb{R}^2$ about L.

(b) T is the projection on L alone the line perpendicular to L.

Sol: (a) We assume $T(x, y) = (\bar{x}, \bar{y})$, then we have $\frac{y+\bar{y}}{2} = m\frac{x+\bar{x}}{2}$, and $(x - \bar{x}) + m(y - \bar{y}) = 0$. Solving the equation we obtain

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ -\frac{2m}{1+m^2} & \frac{1-m^2}{1+m^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence $T = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$

(b) Siimlarly we assume $T(x, y) = (\bar{x}, \bar{y})$, solving $\bar{y} = m\bar{x}$, and $(x - \bar{x}) + m(y - \bar{y}) = 0$, we have

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence $T = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}$

Optional Part

Sec. 2.4

1 Q: Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectively, $T : V \to W$ is linear, and A and B are matrices.

(a) $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$.
(b) T is invertible if and only if T is one-to-one and onto.
(c) $T = \mathsf{L}_A$, where $A = [T]_\alpha^\beta$.
(d) $\mathsf{M}_{2\times 3}(F)$ is isomorphic to F^5.
(e) $\mathsf{P}_n(F)$ is isomorphic to $\mathsf{P}_m(F)$ if and only if $n = m$.
(f) $AB = I$ implies that A and B are invertible.
(g) If A is invertible, then $(A^{-1})^{-1} = A$.
(h) A is invertible if and only if L_A is invertible.
(i) A must be square in order to possess an inverse.

Sol: (a) True.
(b) True.
(c) False.
(d) False.
(e) True.
(f) True.
(g) True.
(h) True.
(i) True.

11 Q: Verify that the transformation in Example 5 is one-to-one.
(Example 5 in textbook) Define

$$T : P_3(R) \to M_{2\times 2}(R) \quad \text{by} \quad T(f) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}.$$

It is easily verified that T is linear. By use of the Lagrange interpolation formula in Section 1.6, it can be shown (compare with Exercise 22) that $T(f) = O$ only when f is the zero polynomial. Thus T is one-to-one (see Exercise 11).

Sol: Let $f_1, f_2 \in P_3(R)$, $T(f_1) = T(f_2)$, it suffice to show $f_1 = f_2$.
Since T is linear, we have

$$T(f_1 - f_2) = T(f_1) - T(f_2) = 0,$$

so

$$(f_1 - f_2)(1) = (f_1 - f_2)(2) = (f_1 - f_2)(3) = (f_1 - f_2)(4) = 0.$$

By interpolation, $f_1 - f_2$ is zero polynomial, so $f_1 = f_2$.

17 Q: Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

Sol: (a) For $T(x_1), T(x_2) \in T(V_0)$, we have

$$\lambda_1 T(x_1) + \lambda_2 T(x_2) = T(\lambda_1 x_1 + \lambda_2 x_2) \in T(V_0), \qquad \forall \lambda_1, \lambda_2 \in F.$$

this is because $\lambda_1 x_1 + \lambda_2 x_2 \in V_0$ since V_0 is a subspace of V. Hence $T(V_0)$ is a subspace of W.

(b) Since T is an isomorphism, $T|_{V_0} : V_0 \to T(V_0)$ is also an isomorphism. Hence by Theorem 2.19 we get $\dim(V_0) = \dim(T(V_0))$.

23 Q: Let V denote the vector space defined in Example 5 of Section 1.2, and let $W = \mathsf{P}(F)$. Define

$$T : V \to W \quad \text{by} \quad T(\sigma) = \sum_{i=0}^{n} \sigma(i) x^i,$$

where n is the largest integer such that $\sigma(n) \neq 0$. Prove that T is an isomorphism.

Sol: To show that T is an isomorphism, we need to prove T is linear, one-to-one and onto. Observe that if n is an non-negative integer such that $\sigma(m) = 0$ for any integer m greater than n, then $T(\sigma) = \sum_{i=0}^{n} \sigma(i) x^i$. Let $\sigma, \tau \in V$ and $c \in F$. Pick a non-negative integer n_σ (resp. n_τ) such that $\sigma(m) = 0$ (resp. $\tau(m) = 0$ for any integer m greater than n_σ (resp. n_τ). Let $n = \max\{n_\sigma, n_\tau\}$. Then for any integer m greater than n, $(\sigma + c\tau)(m) = 0 + c \cdot 0 = 0$. Hence,

$$T(\sigma + c\tau) = \sum_{i=0}^{n} (\sigma + c\tau)(i) x^i = \sum_{i=0}^{n} \sigma(i) x^i + c \sum_{i=0}^{n} \tau(i) x^i = T(\sigma) + cT(\tau).$$

T is thus linear.

Suppose $\sigma \in \mathsf{N}(T)$. Pick a non-negative integer n such that $\forall$ integer m with $m > n$, $\sigma(m) = 0$. Then

$$\sum_{i=0}^{n} \sigma(i) x^i = T(\sigma) = 0.$$

By comparing coefficients, $\sigma(0) = \cdots = \sigma(n) = 0$. Hence, $\sigma(m) = 0$ for any non-negative integer m. T is thus one-to-one.

Let $f \in W$. Write $f(x) = \sum_{i=0}^{n} a_i x^i$, where n is a non-negative integer and $a_0, ..., a_n \in F$. Define $\sigma \in V$ by $\forall$ non-negative integer m,

$$\sigma(m) = \begin{cases} a_m & \text{if } m \leq n; \\ 0 & \text{if } m > n. \end{cases}$$

Hence T is onto. We are done.

24 Q: Let $T : V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping

$$\overline{T} : V/\mathsf{N}(T) \to Z \quad \text{by} \quad \overline{T}(v + \mathsf{N}(T)) = T(v)$$

for any coset $v + \mathsf{N}(T)$ in $V/\mathsf{N}(T)$.

(a) Prove that $\overline{T}$ is well-defined; that is, prove that if $v + \mathsf{N}(T) = v' + \mathsf{N}(T)$, then $T(v) = T(v')$.

(b) Prove that $\overline{T}$ is linear.

(c) Prove that $\overline{T}$ is isomorphism.

(d) Prove that the diagram shown in Figure 2.3 commutes; that is, prove that $T = \overline{T}\eta$.

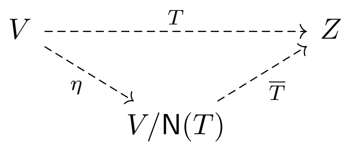

Figure 2.3

Sol: (a) Suppose $v, v' \in V$ and $v + \mathsf{N}(T) = v' + \mathsf{N}(T)$. Let $w = v - v'$. Then $w \in \mathsf{N}(T)$ and hence $T(v) = T(v' + w) = T(v') + T(w) = T(v')$. Therefore, $\overline{T}$ is well-defined.

(b) Let $v, v' \in V$ and $c \in F$. Then

$$\begin{aligned}\overline{T}((v + \mathsf{N}(T)) + c(v' + \mathsf{N}(T))) &= \overline{T}((v + cv') + \mathsf{N}(T)) \\ &= T(v + c'v) = T(v) + cT(v') \\ &= \overline{T}(v + \mathsf{N}(T)) + c\overline{T}(v' + \mathsf{N}(T)).\end{aligned}$$

Therefore, $\overline{T}$ is linear.

(c) By (b), it remains to show that T is one-to-one and onto.
Let $v \in V$ and $v + \mathsf{N}(T) \in \mathsf{N}(\overline{T})$. Then $T(v) = \overline{T}(v + \mathsf{N}(T)) = 0$. In other words, $v \in \mathsf{N}(T)$. Hence $v + \mathsf{N}(T) = \mathsf{N}(T)$. $\overline{T}$ is one-to-one.
Let $z \in Z$. Since T is onto, $\exists v \in V$ such that $\overline{T}(v + \mathsf{N}(T)) = T(v) = z$. Thus, $\overline{T}$ is also onto.
To conclude, $\overline{T}$ is an isomorphism.

(d) Fix $v \in V$. Then

$$\overline{T}(\eta(v)) = \overline{T}(v + \mathsf{N}(T)) = T(v).$$

Therefore, $\overline{T}\eta = T$.

Sec. 2.5

1 Q: Label the following statements as true or false.

(a) Suppose that $\beta = \{x_1, x_2, ..., x_n\}$ and $\beta' = \{x_1', x_2', ..., x_n'\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$.

(b) Every change of coordinate matrix is invertible.

(c) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

(d) The matrices $A, B \in \mathsf{M}_{n\times n}(F)$ are called similar if $B = Q^{\mathrm{t}}AQ$ for some $Q \in \mathsf{M}_{n\times n}(F)$.

(e) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[T]_\beta$ is similar to $[T]_\gamma$.

Sol: (a) False.

(b) True.
(c) True.
(d) False.
(e) True.

11 Q: Let V be a finite-dimensional vector space with ordered bases α, β, and γ.

(a) Prove that if Q and R are the change of coordinate matrices that change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinate matrix that changes α-coordinates into γ-coordinates.

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

Sol: Write $\alpha = \{a_1, ..., a_n\}$, $\beta = \{b_1, ..., b_n\}$ and $\gamma = \{c_1, ..., c_n\}$.

(a) We have

$$a_k = \textstyle\sum_{j=1}^n Q_{jk} b_j \quad \forall k \in \{1, ..., n\};$$
$$b_j = \textstyle\sum_{i=1}^n R_{ij} c_i \quad \forall j \in \{1, ..., n\}.$$

Thus, $\forall k \in \{1, ..., n\}$, $a_k = \sum_{j=1}^n \sum_{i=1}^n R_{ij} Q_{jk} c_i = \sum_{i=1}^n (RQ)_{ik} a_i$. In other words, RQ is the change fo coordinate matrix that changes α-coordinates into γ-coordinates.

(b) $\forall i, j \in \{1, ..., n\}$, let $\delta_{ij} = 1$ if $i = j$; and $\delta_{ij} = 0$ if $i \neq j$. $\forall k \in \{1, ..., n\}$,

$$\sum_{j=1}^n (Q^{-1})_{jk} a_j = \sum_{i=1}^n \sum_{j=1}^n (Q^{-1})_{jk} Q_{ij} b_i = \sum_{i=1}^n \delta_{ik} a_i = b_k.$$

Therefore, Q^{-1} is the change of coordinate matrix that changes β-coordinates into α-coordinates.

13 Q: Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, x_2, ..., x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^n Q_{ij} x_i \quad \text{for} 1 \leq j \leq n,$$

and set $\beta' = \{x'_1, x'_2, ..., x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

Sol: Suppose $c_1, ..., c_n \in F$ and $\sum_{j=1}^n c_j x'_j = \vec{0}$. Then

$$\sum_{i=1}^n \sum_{j=1}^n c_j Q_{ij} x_i = \vec{0}.$$

Since β is linearly independent, we have a system of linear equations

$$\begin{cases} Q_{11} c_1 + \cdots + Q_{1n} c_n &= 0, \\ &\vdots \\ Q_{n1} c_1 + \cdots + Q_{nn} c_n &= 0. \end{cases}$$

As Q is invertible, $c_1 = \cdots = c_n = 0$. Therefore, β' is linearly independent. This also forces that $x_1', ..., x_n'$ are distinct (otherwise, say $x_i' = x_j'$ but $i \neq j$, then $1 \cdot x_i' + (-1) \cdot x_j' = \vec{0}$ which leads to contradiction). As V is of dimension n and the cardinality of β' is also n, β' is a basis for V.