SUGGESTED SOLUTIONS TO HOMEWORK 5

1. COMPULSORY PART

Exercise 1. Let $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $\mathsf{T}(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$. Compute $[\mathsf{T}]_\beta^\gamma$. If $\alpha = \{(1,2),(2,3)\}$, compute $[\mathsf{T}]_\alpha^\gamma$.

Solution. Since

$$\begin{aligned}\mathsf{T}(1,0) &= (1,1,2) = -\frac{1}{3}\cdot(1,1,0) + 0\cdot(0,1,1) + \frac{2}{3}\cdot(2,2,3),\\ \mathsf{T}(0,1) &= (-1,0,1) = -1\cdot(1,1,0) + 1\cdot(0,1,1) + 0\cdot(2,2,3),\end{aligned}$$

therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}.$$

Since

$$\begin{aligned}\mathsf{T}(1,2) &= (-1,1,4) = -\frac{7}{3}\cdot(1,1,0) + 2\cdot(0,1,1) + \frac{2}{3}\cdot(2,2,3),\\ \mathsf{T}(2,3) &= (-1,2,7) = -\frac{11}{3}\cdot(1,1,0) + 3\cdot(0,1,1) + \frac{4}{3}\cdot(2,2,3),\end{aligned}$$

therefore

$$[\mathsf{T}]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}.$$

Exercise 2. Let

$$\begin{aligned}\alpha &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},\\ \beta &= \{1, x, x^2\},\end{aligned}$$

and

$$\gamma = \{1\}.$$

(a) Define $\mathsf{T} : \mathsf{M}_{2\times 2}(\mathbb{F}) \to \mathsf{M}_{2\times 2}(\mathbb{F})$ by $\mathsf{T}(A) = A^t$. Compute $[\mathsf{T}]_\alpha$ and $[\mathsf{T}\begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}]_\alpha$.

(b) Define

$$\mathsf{T} : \mathsf{P}_2(\mathbb{R}) \to \mathsf{M}_{2\times 2}(\mathbb{R}) \text{ by } \mathsf{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[\mathsf{T}]_\beta^\alpha$ and $[\mathsf{T}(4 - 6x + 3x^2)]_\beta^\alpha$.

Solution. (a) Since

$$\mathsf{T}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = 1\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$
$$\mathsf{T}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 1\cdot\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$
$$\mathsf{T}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + 1\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$
$$\mathsf{T}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 1\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$

therefore

$$[\mathsf{T}]_\alpha = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix},$$

and

$$[\mathsf{T}\begin{pmatrix}1 & 4\\-1 & 6\end{pmatrix}]_\alpha = [\begin{pmatrix}1 & -1\\4 & 6\end{pmatrix}]_\alpha = \begin{pmatrix}1\\-1\\4\\6\end{pmatrix}.$$

(b) Since

$$\mathsf{T}(1) = \begin{pmatrix}0 & 2\\0 & 0\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + 2\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$
$$\mathsf{T}(x) = \begin{pmatrix}1 & 2\\0 & 0\end{pmatrix} = 1\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + 2\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$
$$\mathsf{T}(x^2) = \begin{pmatrix}0 & 2\\0 & 2\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + 2\cdot\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + 2\cdot\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$

therefore

$$[\mathsf{T}]_\beta^\alpha = \begin{pmatrix}0 & 1 & 0\\2 & 2 & 2\\0 & 0 & 0\\0 & 0 & 2\end{pmatrix},$$

and

$$[\mathsf{T}(4-6x+3x^2)]_\beta^\alpha = [\begin{pmatrix}-6 & 2\\0 & 6\end{pmatrix}]_\beta^\alpha = \begin{pmatrix}-6\\2\\0\\6\end{pmatrix}.$$

Exercise 3. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, ..., v_n\}$. Define $v_0 = 0$. There exists a linear transformation $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T}(v_j) = v_j + v_{j-1}$ for $j = 1, 2, ..., n$. Compute $[\mathsf{T}]_\beta$.

Solution. Since

$$[\mathsf{T}(v_j)]_\beta = [v_j]_\beta + [v_{j-1}]_\beta,$$

therefore

$$([\mathsf{T}]_\beta)_{ij} = \delta_{ji} + \delta_{j-1,i},$$

for $i, j = 1, 2, ..., n$, where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Exercise 4. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $\mathsf{R}(\mathsf{T}) \cap \mathsf{R}(\mathsf{U}) = \{0\}$, prove that $\{\mathsf{T}, \mathsf{U}\}$ is a linearly independent subset of $\mathcal{L}(\mathsf{V}, \mathsf{W})$.

Solution. Since T and U are nonzero linear transformations, then there exists $v_1, v_2 \in \mathsf{V}$ such that $\mathsf{T}(v_1) \neq 0$ and $\mathsf{U}(v_2) \neq 0$. Assume there exist $c_1, c_2 \in \mathbb{F}$ such that

$$c_1\mathsf{T} + c_2\mathsf{U} = 0_{\mathcal{L}(\mathsf{V},\mathsf{W})},$$

then

$$c_1\mathsf{T}(v_1) + c_2\mathsf{U}(v_1) = 0_{\mathsf{W}}, \quad c_1\mathsf{T}(v_2) + c_2\mathsf{U}(v_2) = 0_{\mathsf{W}},$$

which implies that

$$\mathsf{T}(c_1v_1) = \mathsf{U}(-c_2v_1), \quad \mathsf{T}(c_1v_2) = \mathsf{U}(-c_2v_2).$$

Since $\mathsf{R}(\mathsf{T}) \cap \mathsf{R}(\mathsf{U}) = \{0\}$, therefore

$$\mathsf{T}(c_1v_1) = 0_{\mathsf{W}}, \quad \mathsf{U}(-c_2v_2) = 0_{\mathsf{W}},$$

which implies that

$$c_1 = c_2 = 0,$$

therefore T and U are linearly independent.

Exercise 5. Let $\mathsf{V} = \mathsf{P}(\mathbb{R})$, and for $j \geq 1$ define $\mathsf{T}_j(f(x)) = f^{(j)}(x)$, where $f^{(j)}(x)$ is the jth derivative of $f(x)$. Prove that the set $\{\mathsf{T}_1, \mathsf{T}_2, ..., \mathsf{T}_n\}$ is a linearly independent subset of $\mathcal{L}(\mathsf{V})$ for any positive integer n.

Solution. Let $\alpha_1, ..., \alpha_n \in F$ such that

$$\sum_{i=1}^{n} \alpha_i\mathsf{T}_i = 0,$$

then

$$\sum_{i=1}^{n} \alpha_i\mathsf{T}_i(x) = \alpha_1 = 0,$$

$$\sum_{i=1}^{n} \alpha_i\mathsf{T}_i(x^2) = \alpha_1 \cdot 2x + \alpha_2 \cdot 2 = 0,$$

$$\vdots$$

$$\sum_{i=1}^{n} \alpha_i\mathsf{T}_i(x^n) = \alpha_1 \cdot nx^{n-1} + \alpha_2 \cdot n(n-1)x^{n-2} + \cdots + \alpha_n \cdot n! = 0,$$

therefore

$$\alpha_1 = \alpha_2 = \cdots = a_n = 0,$$

which implies $\{\mathsf{T}_1, \mathsf{T}_2, ..., \mathsf{T}_n\}$ is a linearly independent subset.

Exercise 6. Let $g(x) = 3 + x$. Let $\mathsf{T} : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ and $\mathsf{U} : \mathsf{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$\mathsf{T}(f(x)) = f'(x)g(x) + 2f(x) \text{ and } \mathsf{U}(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases of $\mathsf{P}_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(a) Compute $[\mathsf{U}]_\beta^\gamma$, $[\mathsf{T}]_\beta$, and $[\mathsf{UT}]_\beta^\gamma$ directly.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[\mathsf{U}(h(x))]_\gamma$.

Solution. (a) Since

$$\begin{aligned}
\mathsf{U}(1) &= (1, 0, 1) = 1 \cdot (1, 0, 0) + 0 \cdot (0, 1, 0) + 1 \cdot (0, 0, 1), \\
\mathsf{U}(x) &= (1, 0, -1) = 1 \cdot (1, 0, 0) + 0 \cdot (0, 1, 0) + (-1) \cdot (0, 0, 1), \\
\mathsf{U}(x^2) &= (0, 1, 0) = 0 \cdot (1, 0, 0) + 1 \cdot (0, 1, 0) + 0 \cdot (0, 0, 1),
\end{aligned}$$

then

$$[\mathsf{U}]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Since

$$\begin{aligned}
\mathsf{T}(1) &= 2 = 2 \cdot 1 + 0 \cdot x + 0 \cdot x^2, \\
\mathsf{T}(x) &= 3 + 3x = 3 \cdot 1 + 3 \cdot x + 0 \cdot x^2, \\
\mathsf{T}(x^2) &= 6x + 4x^2 = 0 \cdot 1 + 6 \cdot x + 4 \cdot x^2,
\end{aligned}$$

then

$$[\mathsf{T}]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}.$$

Since

$$\begin{aligned}
\mathsf{UT}(1) &= (2, 0, 2) = 2 \cdot (1, 0, 0) + 0 \cdot (0, 1, 0) + 2 \cdot (0, 0, 1), \\
\mathsf{UT}(x) &= (6, 0, 0) = 6 \cdot (1, 0, 0) + 0 \cdot (0, 1, 0) + 0 \cdot (0, 0, 1), \\
\mathsf{UT}(x^2) &= (6, 4, -6) = 6 \cdot (1, 0, 0) + 4 \cdot (0, 1, 0) + (-6) \cdot (0, 0, 1),
\end{aligned}$$

then

$$[\mathsf{UT}]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.$$

(b) Since

$$h(x) = 3 \cdot 1 + (-2) \cdot x + 1 \cdot x^2,$$

then

$$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

Since

$$\mathsf{U}(h(x)) = (1, 1, 5) = 1 \cdot (1, 0, 0) + 1 \cdot (0, 1, 0) + 5 \cdot (0, 0, 1),$$

then

$$[\mathsf{U}(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$$

Exercise 7. Let T, W, and Z be vector spaces, and let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ and $\mathsf{U} : \mathsf{W} \to \mathsf{Z}$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one ?

(b) Prove that if UT is onto, then U is onto. Must T also be onto ?

(c) Prove that if U and T are one-to-one and onto, then UT is also.

Solution. (a) Assume that there exists $v \in \mathsf{V}$ such that

$$\mathsf{T}(v) = 0_{\mathsf{W}},$$

then

$$\mathsf{UT}(v) = 0_{\mathsf{Z}},$$

since UT is one-to-one, therefore

$$v = 0_{\mathsf{V}},$$

which implies that T is one-to-one.

However, U is not necessarily one-to-one. Indeed, consider

$$\begin{aligned} \mathsf{U} :&\mathbb{R}^2 \to \mathbb{R} \\ (x, y) &\mapsto x, \end{aligned} \qquad \begin{aligned} \mathsf{T} :&\mathbb{R} \to \mathbb{R}^2 \\ x &\mapsto (x, 0), \end{aligned}$$

then T is one-to-one but U is not one-to-one.

(b) For arbitrary $z \in \mathsf{Z}$, since UT is onto, there exists $v \in \mathsf{V}$ such that

$$\mathsf{UT}(v) = z,$$

therefore

$$\mathsf{U}(\mathsf{T}(v)) = z,$$

which implies that U is onto.

However, T is not necessarily onto. Indeed, consider

$$\begin{aligned} \mathsf{U} :&\mathbb{R}^2 \to \mathbb{R} \\ (x, y) &\mapsto x, \end{aligned} \qquad \begin{aligned} \mathsf{T} :&\mathbb{R} \to \mathbb{R}^2 \\ x &\mapsto (x, 0), \end{aligned}$$

then U is onto but T is not onto.

(c) To prove UT is one-to-one, assume that there exists $v \in \mathsf{V}$ such that

$$\mathsf{UT}(v) = 0_{\mathsf{Z}},$$

since U is one-to-one, therefore

$$\mathsf{T}(v) = 0_{\mathsf{W}},$$

since T is one-to-one, therefore

$$v = 0_{\mathsf{V}},$$

which implies that UT is one-to-one.

To prove that UT is onto. For arbitrary $z \in \mathsf{Z}$, since U is onto, there exists $w \in \mathsf{W}$ such that

$$\mathsf{U}(w) = z,$$

since T is onto, there exists $v \in \mathsf{V}$ such that

$$\mathsf{T}(v) = w,$$

which implies that

$$\mathsf{UT}(v) = z,$$

therefore UT is onto.

2. OPTIONAL PART

Exercise 8. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $\mathsf{T}, \mathsf{U} : \mathsf{V} \to \mathsf{W}$ are linear transformations.

(a) For any scalar a, $a\mathsf{T} + \mathsf{U}$is a linear transformation from V to W.
(b) $[\mathsf{T}]_\beta^\gamma = [\mathsf{U}]_\beta^\gamma$ implies that $\mathsf{T} = \mathsf{U}$.
(c) If $m = \dim(\mathsf{V})$ and $n = \dim(\mathsf{W})$, then $[\mathsf{T}]_\beta^\gamma$ is an $m \times n$ matrix.
(d) $[\mathsf{T} + \mathsf{U}]_\beta^\gamma = [\mathsf{T}]_\beta^\gamma + [\mathsf{U}]_\beta^\gamma$.
(e) $\mathcal{L}(\mathsf{V}, \mathsf{W})$ is a vector space.
(f) $\mathcal{L}(\mathsf{V}, \mathsf{W}) = \mathcal{L}(\mathsf{W}, \mathsf{V})$.

Solution. (a) True.
(b) True.
(c) False. Indeed, $[\mathsf{T}]_\beta^\gamma$ is a $n \times m$ matrix.
(d) True.
(e) True.
(f) False. Indeed, consider $\mathsf{V} = \mathbb{R}$ and $\mathsf{W} = \mathbb{R}^2$.

Exercise 9. Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^m$, compute $[\mathsf{T}]_\beta^\gamma$.

(a) $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.
(b) $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$.
(c) $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$.
(d) $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_2 + a_3, -a_1 + 4a_2 + 5a_3, a_1 + a_3)$.
(e) $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathsf{T}(a_1, a_2, ..., a_n) = (a_1, a_1, ..., a_1)$.
(f) $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathsf{T}(a_1, a_2, ..., a_n) = (a_n, a_{n-1}, ..., a_1)$.
(g) $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}$ defined by $\mathsf{T}(a_1, a_2, ..., a_n) = a_1 + a_n$.

Solution. (a) Since

$$\mathsf{T}(1,0) = (2,3,1) = 2 \cdot (1,0,0) + 3 \cdot (0,1,0) + 1 \cdot (0,0,1),$$
$$\mathsf{T}(0,1) = (-1,4,0) = -1 \cdot (1,0,0) + 4 \cdot (0,1,0) + 0 \cdot (0,0,1),$$

therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}.$$

(b) Since

$$\mathsf{T}(1,0,0) = (2,1) = 2 \cdot (1,0) + 1 \cdot (0,1),$$
$$\mathsf{T}(0,1,0) = (3,0) = 3 \cdot (1,0) + 0 \cdot (0,1),$$
$$\mathsf{T}(0,0,1) = (-1,1) = -1 \cdot (1,0) + 1 \cdot (0,1),$$

therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

(c) Since

$$\mathsf{T}(1,0,0) = 2,$$
$$\mathsf{T}(0,1,0) = 1,$$
$$\mathsf{T}(0,0,1) = -3,$$

therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}.$$

(d) Since

$$\begin{aligned}
\mathsf{T}(1,0,0) &= (0,-1,1) = 0\cdot(1,0,0) + (-1)\cdot(0,1,0) + 1\cdot(0,0,1),\\
\mathsf{T}(0,1,0) &= (2,4,0) = 2\cdot(1,0,0) + 4\cdot(0,1,0) + 0\cdot(0,0,1),\\
\mathsf{T}(0,0,1) &= (1,5,1) = 1\cdot(1,0,0) + 5\cdot(0,1,0) + 1\cdot(0,0,1),
\end{aligned}$$

therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{pmatrix}.$$

(e) Since

$$\begin{aligned}
\mathsf{T}(1,0,...,0) &= (1,1,...,1) = 1\cdot(1,0,...,0) + \cdots + 1\cdot(0,0,...,1),\\
\mathsf{T}(0,1,...,0) &= (0,0,...,0) = 0\cdot(1,0,...,0) + \cdots + 0\cdot(0,0,...,1),\\
&\vdots\\
\mathsf{T}(0,0,...,1) &= (0,0,...,0) = 0\cdot(1,0,...,0) + \cdots + 0\cdot(0,0,...,1),
\end{aligned}$$

therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

(f) Since

$$\begin{aligned}
\mathsf{T}(1,0,...,0) &= (0,...,0,1) = 0\cdot(1,0,...,0) + \cdots + 0\cdot(0,...,1,0) + 1\cdot(0,0,...,1),\\
\mathsf{T}(0,1,...,0) &= (0,...,1,0) = 0\cdot(1,0,...,0) + \cdots + 1\cdot(0,...,1,0) + 0\cdot(0,0,...,1),\\
&\vdots\\
\mathsf{T}(0,0,...,1) &= (1,0,...,0) = 1\cdot(1,0,...,0) + \cdots + 0\cdot(0,...,1,0) + 0\cdot(0,0,...,1),
\end{aligned}$$

therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

(g) Since

$$\begin{aligned}
\mathsf{T}(1,0,...,0,0) &= 1,\\
\mathsf{T}(0,1,...,0,0) &= 0,\\
&\vdots\\
\mathsf{T}(0,0,...,1,0) &= 0,\\
\mathsf{T}(0,0,...,0,1) &= 1,
\end{aligned}$$

therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Exercise 10. Let V be an n-dimensional vector space, and let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear transformation. Suppose that W is T-invariant subspace of V having dimension k. Show that there is a basis β for V such that $[\mathsf{T}]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.

Solution. Let $\alpha = \{w_1, ..., w_k\}$ be an ordered basis for W. Then by Replacement theorem, there exists a linearly independent subset $\alpha' = \{w_1', ..., w_{n-k}'\}$ in V such that $\beta := \alpha \cup \alpha'$ is a basis for V. We claim that

$$[\mathsf{T}]_\beta = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

It suffices to prove that $([\mathsf{T}]_\beta)_{ij} = 0$ for $k+1 \le i \le n$, $1 \le j \le k$. Indeed, since W is T-invariant, then $\mathsf{T}(w_j) \in \mathsf{W}$ for $1 \le j \le k$, which implies

$$([\mathsf{T}(w_j)]_\beta)_i = 0,$$

for $k+1 \le i \le n$, $1 \le j \le k$.

Exercise 11. Let V and W be vector spaces such that $\dim(\mathsf{V}) = \dim(\mathsf{W})$, and let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[\mathsf{T}]_\beta^\gamma$ is a diagonal matrix.

Solution. By the dimension theorem,

$$\dim \mathsf{N}(\mathsf{T}) + \dim \mathsf{R}(\mathsf{T}) = \dim \mathsf{W}.$$

And $\dim(\mathsf{V}) = \dim(\mathsf{W})$. Then let $\alpha_\mathsf{V} = \{v_1, ..., v_n\}$ be an ordered basis of $\mathsf{N}(\mathsf{T})$ and $\alpha_\mathsf{W} = \{w_1, ..., w_m\}$ be an ordered basis of $\mathsf{R}(\mathsf{T})$, by the Replacement theorem, there exists a linearly independent subset $\alpha_\mathsf{W}' = \{w_1', ..., w_n'\}$ such that $\gamma := \alpha_\mathsf{W}' \cup \alpha_\mathsf{W}$ is a basis of W. Moreover, denote

$$v_i' := \mathsf{T}^{-1}(w_i),$$

for $i = 1, ..., m$, and

$$\alpha_\mathsf{V}' := \{v_1', ..., v_m'\}.$$

We claim that for $\beta := \alpha_\mathsf{V}' \cup \alpha_\mathsf{V}$ and $\gamma = \alpha_\mathsf{W} \cup \alpha_\mathsf{W}'$, $[\mathsf{T}]_\beta^\gamma$ is a diagonal matrix. Indeed,

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} I_m & O_n \\ O_n & O_m \end{pmatrix},$$

where O_n and O_m are n-th and m-th zero matrices respectively.

Exercise 12. Label the following statements as true or false. In each part, V, W, and Z denote vector spaces with ordered (finite) bases α, β, and γ, respectively; $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ and $\mathsf{U} : \mathsf{W} \to \mathsf{Z}$ denote linear transformations; and A and B denote matrices.

(a) $[\mathsf{UT}]_\alpha^\gamma = [\mathsf{T}]_\beta^\gamma[\mathsf{U}]_\beta^\gamma$.
(b) $[\mathsf{T}(v)]_\beta = [\mathsf{T}]_\alpha^\beta[v]_\alpha$ for all $v \in \mathsf{V}$.
(c) $[\mathsf{U}(w)]_\beta = [\mathsf{U}]_\alpha^\beta[w]_\beta$ for all $w \in \mathsf{W}$.
(d) $[\mathsf{I_V}]_\alpha = I$.
(e) $[\mathsf{T}^2]_\alpha^\beta = ([\mathsf{T}]_\alpha^\beta)^2$.
(f) $A^2 = I$ implies that $A = I$ or $A = -I$.
(g) $\mathsf{T} = \mathsf{L}_A$ for some matrix A.
(h) $A^2 = O$ implies that $A = O$, where O denotes the zero matrix.
(i) $\mathsf{L}_{A+B} = \mathsf{L}_A + \mathsf{L}_B$.
(j) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$.

Solution. (a) False. Indeed, $[\mathsf{UT}]_\alpha^\gamma = [\mathsf{U}]_\beta^\gamma[\mathsf{T}]_\alpha^\beta$.
(b) True.
(c) False. Indeed, $[\mathsf{U}(w)]_\beta = [\mathsf{U}]_\beta^\gamma[w]_\beta$.
(d) True.
(e) False. Indeed, it makes sense only $\alpha = \beta$.
(f) False. Indeed, consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $A^2 = I$ but $A \neq I$ and $A \neq -I$.
(g) False. Indeed, $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ but $\mathsf{L}_A : \mathbb{F}^m \to \mathbb{F}^n$.
(h) False. Indeed,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then $A^2 = I$ but $A \neq O$.
(i) True.
(j) True.

Exercise 13. Let V be a vector space, and let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be linear. Prove that $\mathsf{T}^2 = \mathsf{T}_0$ if and only if $\mathsf{R}(\mathsf{T}) \subset \mathsf{N}(\mathsf{T})$.

Solution. $\Rightarrow$: Let $y \in \mathsf{R}(\mathsf{T})$, then there exists $x \in \mathsf{V}$ such that

$$y = \mathsf{T}(x),$$

then

$$\mathsf{T}(y) = \mathsf{T}_0(x) = 0,$$

which implies that $y \in \mathsf{N}(\mathsf{T})$, by the arbitrary choice of y, we have $\mathsf{R}(\mathsf{T}) \subset \mathsf{N}(\mathsf{T})$.
$\Leftarrow$: Let $x \in \mathsf{V}$, then $\mathsf{T}(x) \in \mathsf{R}(\mathsf{T})$, which implies $\mathsf{T}(x) \in \mathsf{N}(\mathsf{T})$, therefore

$$\mathsf{T}^2(x) = \mathsf{T}(\mathsf{T}(x)) = 0,$$

by the arbitrary choice of x, we have $\mathsf{T}^2 = \mathsf{T}_0$.

Exercise 14. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$tr(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $tr(AB) = tr(BA)$ and $tr(A) = tr(A^t)$.

Solution. To prove $tr(AB) = tr(BA)$, it suffices to note that

$$tr(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} = tr(BA).$$

To prove $tr(A) = tr(A^t)$, it suffices to note that $A_{ii} = (A^t)_{ii}$ for $1 \leq i \leq n$.

Exercise 15. Let V be a finite-dimensional vector space, and let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be linear.

(a) If $rank(\mathsf{T}) = rank(\mathsf{T}^2)$, prove that $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$. Deduce that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$.

(b) Prove that $\mathsf{V} = \mathsf{R}(\mathsf{T}^k) \oplus \mathsf{N}(\mathsf{T}^k)$ for some positive integer k.

Solution. (a) Let $y_0 \in \mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T})$, then $\mathsf{T}(y_0) = 0_\mathsf{V}$ and there exists $x_0 \in \mathsf{V}$ such that

$$y_0 = \mathsf{T}(x_0),$$

therefore

$$\mathsf{T}^2(x_0) = 0_\mathsf{V},$$

which implies that $x_0 \in \mathsf{N}(\mathsf{T}^2)$. By the Dimension theorem,

$$\dim \mathsf{N}(\mathsf{T}) = \dim \mathsf{N}(\mathsf{T}^2).$$

Then by the Replacement theorem, there exists a linearly independent subset $\alpha = \{y_1, ..., y_{n-1}\}$ such that $\{y_0, y_1, ..., y_{n-1}\}$ is a basis for $\mathsf{N}(\mathsf{T})$. Since for arbitrary $y_i \in \{y_0, y_1, ..., y_{n-1}\}$, $0 \leq i \leq n-1$,

$$\mathsf{T}^2(y_i) = \mathsf{T}(0_\mathsf{V}) = 0_\mathsf{V},$$

which implies that $\{y_0, y_1, ..., y_{n-1}\}$ is also a basis for $\mathsf{N}(\mathsf{T}^2)$. Therefore there exists $c_0, c_1, ..., c_{n-1}$ such that

$$x_0 = \sum_{i=0}^{n-1} c_i y_i,$$

then

$$\mathsf{T}(x_0) = \sum_{i=0}^{n-1} c_i \mathsf{T}(y_i) = 0_\mathsf{V},$$

which implies that $y_0 = 0_\mathsf{V}$. Therefore we have $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$.

By the Dimension theorem, we have

$$\dim \mathsf{N}(\mathsf{T}) + \dim \mathsf{R}(\mathsf{T}) = \dim \mathsf{V}.$$

Moreover, $\mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})$ are two subspaces of V, therefore $\mathsf{V} = \mathsf{N}(\mathsf{T}) \oplus \mathsf{R}(\mathsf{T})$.

(b) It suffices to prove that there exists $k_0 \in \mathbb{N}$ such that $rank(\mathsf{T}^{k_0}) = rank(\mathsf{T}^{k_0+1})$. Indeed, since

$$rank(\mathsf{T}^k) \geq rank(\mathsf{T}^{k+1}) \geq 0,$$

which implies that $rank(\mathsf{T}^k)$ is non-increasing as k goes to infinity and bounded below by 0. Therefore there exists a finite $k_0 \in \mathbb{N}$ such that

$$rank(\mathsf{T}^{k_0}) = rank(\mathsf{T}^{k_0+1}).$$

Exercise 16. Let V be a vector space. Determine all linear transformations $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T}^2 = \mathsf{T}$.

Solution. We claim that $\mathsf{T}^2 = \mathsf{T}$ if and only if T is the projection on a subspace.

$\Rightarrow$: We prove that T is the projection on $\mathsf{R}(\mathsf{T})$. Indeed, for arbitrary $y \in \mathsf{R}(\mathsf{T})$, then there exists $x \in \mathsf{V}$ such that

$$y = \mathsf{T}(x),$$

then

$$\mathsf{T}(y) = \mathsf{T}^2(x) = y,$$

by the arbitrary choice of y, we have T is the projection on $\mathsf{R}(\mathsf{T})$.

$\Leftarrow$: Assume that T is the projection on a subspace W of V, then for arbitrary $x \in \mathsf{V}$, since $\mathsf{T}(x) \in \mathsf{W}$, we have$\mathsf{T}^2(x) = \mathsf{T}(x)$, which implies that $\mathsf{T}^2 = \mathsf{T}$.