SUGGESTED SOLUTIONS TO HOMEWORK 3

1. Compulsory part

Exercise 1. Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u+v, au\}$ and $\{au, av\}$ are also bases for V.

Solution. To show that $\{u+v, au\}$ is a base for V, it suffices to prove that $u+v$ and au are linearly independent. Indeed, suppose there exist $\alpha, \beta \in F$ such that

$$\alpha \cdot (u+v) + \beta \cdot au = 0,$$

then

$$\begin{cases} \alpha + a\beta = 0, \\ \alpha = 0, \end{cases}$$

which implies that $\alpha = \beta = 0$.

Similarly, to show that $\{au, av\}$ is a base for V, it suffices to prove that au and av are linearly independent. Indeed, suppose there exist α, $\beta \in F$ such that

$$\alpha \cdot au + \beta \cdot av = 0,$$

then

$$\begin{cases} a\alpha = 0, \\ a\beta = 0, \end{cases}$$

which implies that $\alpha = \beta = 0$.

Exercise 2. Le u, v, w be distinct vectors of a vector a space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

Solution. It suffices to prove that $u+v+w$, $v+w$ and w are linearly independent. Indeed, suppose there exist α, β, $\gamma \in F$ such that

$$\alpha(u+v+w) + \beta(v+w) + \gamma w = 0,$$

then

$$\begin{cases} \alpha = 0, \\ \alpha + \beta = 0, \\ \alpha + \beta + \gamma = 0, \end{cases}$$

which implies that

Exercise 3. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $\mathsf{M}_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

Solution. The basis for W is $\{E_{ij}\}_{i\neq j, 1\leq i,j\leq n} \cup \{E_{11} - E_{ii}\}_{2\leq i\leq n}$. Therefore $\dim(\mathsf{W}) = n^2 - 1$.

Exercise 4. Prove that a vector space is infinite-dimensional if and only if it contains an infinite linearly independent subset.

Solution. $\Rightarrow$: Let V be a infinite-dimensional vector space. Then there exists $0_{\mathsf{V}} \neq v_1 \in \mathsf{V}$, by Replacement theorem, inductively, we can pick $v_{k+1} \in \mathsf{V}$ such that $v_{k+1} \notin \text{span}(\{v_1, v_2, ..., v_k\}) \subsetneq \mathsf{V}$. Since V is infinite dimensional, and by our choice of v_k, we can obtain a set of infinitely many linear independent vector $\{v_k\}_{k\geq 1}$.

$\Leftarrow$: Let S be the infinite linearly independent subset of a vector space V. Then $\text{span}(S)$ is a subspace of V with infinite linearly independent vectors, which implies that V is infinite-dimensional.

Exercise 5. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Determine necessary and sufficient conditions on W_1 and W_2 so that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$.

Solution. We claim that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$ if and only if $\mathsf{W}_1 \subset \mathsf{W}_2$.

$\Rightarrow$: Let $\{w_i\}_{1\leq i\leq n}$ be the basis of $\mathsf{W}_1 \cap \mathsf{W}_2$ where $n = \dim(\mathsf{W}_1 \cap \mathsf{W}_2)$. Since $\mathsf{W}_1 \cap \mathsf{W}_2$ is a subspace of W_1, then by Replacement theorem, there exists a linearly independent set $\{w'_i\}_{1\leq i\leq m}$ such that $\{w_i\}_{1\leq i\leq n} \cup \{w'_i\}_{1\leq i\leq m}$ is a basis of W_1, moreover, $m + n = \dim(\mathsf{W}_1)$, which implies that $m = 0$, then $\{w_i\}_{1\leq i\leq n}$ is also a basis of W_1, therefore $\mathsf{W}_1 \cap \mathsf{W}_2 = \mathsf{W}_1$, or equivalently, $\mathsf{W}_1 \subset \mathsf{W}_2$.

$\Leftarrow$: Since $\mathsf{W}_1 \subset \mathsf{W}_2$, then $\mathsf{W}_1 \cap \mathsf{W}_2 = \mathsf{W}_1$ which implies that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$.

Exercise 6. Let $v_1, v_2, ..., v_k, v$ be vectors in a vector space V, and define $\mathsf{W}_1 = \text{span}(\{v_1, v_2, ..., v_k\})$, and $\mathsf{W}_2 = \text{span}(\{v_1, v_2, ..., v_k, v\})$.

(a) Find necessary and sufficient conditions on v such that $\dim(\mathsf{W}_1) = \dim(\mathsf{W}_2)$.

(b) State and prove a relationship involving $\dim(\mathsf{W}_1)$ and $\dim(\mathsf{W}_2)$ in the case that $\dim(\mathsf{W}_1) \neq \dim(\mathsf{W}_2)$.

Solution. (a) We claim that $\dim(\mathsf{W}_1) = \dim(\mathsf{W}_2)$ if and only if $v \in \mathsf{W}_1$.

$\Rightarrow$: Let $\{w_i\}_{1\leq i\leq n}$ be a basis of W_1 where $n = \dim(\mathsf{W}_1)$. Since W_1 is subspace of W_2, then by Replacement theorem, there exists a linearly independent $\{w'_i\}_{1\leq i\leq m}$ such that $\{w_i\}_{1\leq i\leq n} \cup \{w'_i\}_{1\leq i\leq m}$ is a basis of W_2, moreover, $m + n = \dim(\mathsf{W}_2)$, which implies that $m = 0$, then $\{w_i\}_{1\leq i\leq n}$ is also a basis of W_2, therefore $v \in \text{span}(\{w_i\}_{1\leq i\leq n}) = \mathsf{W}_1$.

$\Leftarrow$: Let $\{w_i\}_{1\leq i\leq n}$ be a basis of W_1 where $n = \dim(\mathsf{W}_1)$. Since $v \in \mathsf{W}_1$, therefore $\{w_i\}_{1\leq i\leq n}$ is also a basis of $\dim(\mathsf{W}_2)$ which implies that $\dim(\mathsf{W}_1) = \dim(\mathsf{W}_2)$.

(b) Since $\mathsf{W}_1 \subset \mathsf{W}_2$, then $\dim(\mathsf{W}_1) < \dim(\mathsf{W}_2)$.

Exercise 7. For a fixed $a \in \mathbb{R}$, determine the dimension of the subspace of $\mathsf{P}_n(\mathbb{R})$ defined by $\{f \in \mathsf{P}_n(\mathbb{R}) : f(a) = 0\}$.

Solution. One basis of $\{f \in \mathsf{P}_n(\mathbb{R}) : f(a) = 0\}$ is $\{x - a, x^2 - a^2, ..., x^n - a^n\}$. Therefore $\dim(\mathsf{P}_n(\mathbb{R})) = n$.

Exercise 8. Let V be a finite-dimensional vector space over $\mathbb{C}$ with dimension n. Prove that if V is now regarded as a vector space over $\mathbb{R}$, then $\dim \mathsf{V} = 2n$.

Solution. We claim that $\{e_i\}_{1\leq i\leq n} \cup \{\mathbf{i}e_i\}_{1\leq i\leq n}$ is a basis of V over $\mathbb{R}$ where $e_i := (0, ..., \overset{\text{i-th}}{1}, ...0)$ and $\mathbf{i}$ is the imaginary unit. Indeed, suppose there exist $\alpha_i, \beta_j \in \mathbb{R}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$ such that

$$\sum_{i=1}^{n} \alpha e_i + \sum_{j=1}^{n} \beta_j \mathbf{i} e_j = 0,$$

then

$$\begin{cases} \alpha_1 + \beta_1 \mathbf{i} = 0, \\ \alpha_2 + \beta_2 \mathbf{i} = 0, \\ \qquad \vdots \\ \alpha_n + \beta_n \mathbf{i} = 0, \end{cases}$$

which implies that $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = 0$.

Exercise 9. Let

$$\mathsf{V} = \mathsf{M}_{2\times 2}(F), \quad \mathsf{W}_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in \mathsf{V} : a, b, c \in F \right\},$$

and

$$\mathsf{W}_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in \mathsf{V} : a, b \in F \right\}.$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, $\mathsf{W}_1 + \mathsf{W}_2$, and $\mathsf{W}_1 \cap \mathsf{W}_2$.

Solution. Let us prove that W_1 and W_2 are subspaces of V. It is straightforward to verify that W_1 and W_2 are closed under vector addition, scalar addition and scalar multiplication with the commutative property, the associative property and the distributive property. Moreover,

$$0_{\mathsf{W}} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is a zero element in W_1 and W_2. $\dim(\mathsf{W}_1) = 3$, $\dim(\mathsf{W}_2) = 2$, $\dim(\mathsf{W}_1 + \mathsf{W}_2) = 4$, $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = 1$. For arbitrary $a, b, c \in F$, Let us denote

$$w_1 := \begin{pmatrix} a & b \\ c & a \end{pmatrix}, \quad w_2 := \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix},$$

then $w_1 \in \mathsf{W}_1$ and $w_2 \in \mathsf{W}_2$. Since $-w_1 \in \mathsf{W}_1$ and $-w_2 \in \mathsf{W}_2$, moreover, $w_1 + (-w_1) = 0_{\mathsf{W}}$ and $w_2 + (-w_2) = 0_{\mathsf{W}}$, therefore $-w_1$ and $-w_2$ are the additive inverse of w_1 and w_2 respectively. In addition, we also have $1w_1 = w_1$ and $1w_2 = w_2$.

Since one basis of W_1 is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\},$$

Indeed, it is straightforward to justify that vectors in the above set are linearly independent and generate W_1, therefore $\dim(\mathsf{W}_1) = 3$.

Since one basis of W_2 is

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

Indeed, it is straightforward to justify that vectors in the above set are linearly independent and generate W_2, therefore $\dim(\mathsf{W}_2) = 2$.

Since one basis of $\mathsf{W}_1 + \mathsf{W}_2$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

Indeed, it is straightforward to justify that vectors in the above set are linearly independent and generate $\mathsf{W}_1 + \mathsf{W}_2$, therefore $\dim(\mathsf{W}_1 + \mathsf{W}_2) = 4$.

Since one basis of $\mathsf{W}_1 \cap \mathsf{W}_2$ is

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

Indeed, it is straightforward to justify that vectors in the above set are linearly independent and generate $\mathsf{W}_1 \cap \mathsf{W}_2$, therefore $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = 1$.

2. OPTIONAL PART

Exercise 10. Label the following statements as true or false.

(a) The zero vector space has no basis.

(b) Every vector space that is generated by a finite set has a basis.

(c) Every vector space has a finite basis.

(d) A vector space cannot have more than one basis.

(e) If a vector space has a finite basis, then the number of vectors in every basis is the same.

(f) The dimension of $\mathsf{P}_n(F)$ is n.

(g) The dimension of $\mathsf{M}_{m\times n}(F)$ is $m+n$.

(h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.

(i) If S generate the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.

(j) Every subspaces of a finite-dimensional space is finite-dimensional.

(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n.

(l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

Solution. (a) False. Indeed, the empty set is the basis.

(b) True.

(c) False. Indeed, $\mathsf{P}(F)$ does not have a finite basis.

(d) False. Indeed, for $\mathbb{R}^2$, $\{(1,0),(0,1)\}$ and $\{(1,1),(1,-1)\}$ are two bases of $\mathbb{R}^2$.

(e) True.

(f) False. Indeed, $\dim(\mathsf{P}_n(F)) = n+1$.

(g) False. Indeed, $\dim(\mathsf{M}_{m\times n}(F)) = mn$.

(h) True.

(i) False. Indeed, let $\mathsf{V} = \mathbb{R}$ and $S = \{1,2\}$, then $4 = 2\times 1 + 2 = 2\times 2$.

(j) True.

(k) True.

(l) True.

Exercise 11. Determine which of the following sets are bases for $\mathsf{P}_2(\mathbb{R})$.

(a) $\{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$

(b) $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$

(c) $\{1 - 2x - 2x^2, -2 + 3x - x^2, 1 - x + 6x^2\}$

(d) $\{-1 + 2x + 4x^2, 3 - 4x - 10x^2, -2 - 5x - 6x^2\}$

(e) $\{1 + 2x - x^2, 4 - 2x + x^2, -1 + 18x - 9x^2\}$

Solution. (a) No. Indeed, $1 - 2x + 4x^2 = 5\times(-1 - x + 2x^2) + 3\times(2 + x - 2x^2)$.

(b) Yes.

(c) Yes.
(d) Yes.
(e) No. Indeed, $-1+18x-9x^2 = 7\times(1+2x-x^2) - 2\times(4-2x+x^2)$.

Exercise 12. Give three different bases for F^2 and for $\mathsf{M}_{2\times 2}(F)$.

Solution. For F^2,

$$\{(1,0),(0,1)\},\quad \{(-1,0),(0,1)\},\quad \{(1,0),(0,-1)\}$$

are three bases.

For $\mathsf{M}_{2\times 2}(F)$,

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\right\},$$

are three bases.

Exercise 13. (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $\mathsf{W}_1+\mathsf{W}_2$ is finite-dimensional, and $\dim(\mathsf{W}_1+\mathsf{W}_2) = \dim(\mathsf{W}_1)+\dim(\mathsf{W}_2)-\dim(\mathsf{W}_1\cap\mathsf{W}_2)$.

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $\mathsf{V} = \mathsf{W}_1+\mathsf{W}_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(\mathsf{V}) = \dim(\mathsf{W}_1)+\dim(\mathsf{W}_2)$.

Solution. (a) Let $\{w_i\}_{1\le i\le n}$ be a basis of $\mathsf{W}_1\cap\mathsf{W}_2$ where $n=\dim(\mathsf{W}_1\cap\mathsf{W}_2)$. Since $\mathsf{W}_1\cap\mathsf{W}_2\subset\mathsf{W}_1$, then by Replacement theorem, there exists a linearly independent set $\{w'_i\}_{1\le i\le n_1}$ such that $\{w_i\}_{1\le i\le n}\cup\{w'_i\}_{1\le i\le n_1}$ is a basis of W_1 and $n+n_1 = \dim(\mathsf{W}_1)$. Similarly, since $\mathsf{W}_1\cap\mathsf{W}_2\subset\mathsf{W}_2$, then by Replacement theorem, there exists a linearly independent set $\{w''_i\}_{1\le i\le n_2}$ such that $\{w_i\}_{1\le i\le n}\cup\{w''_i\}_{1\le i\le n_2}$ is a basis of W_2 and $n+n_2=\dim(\mathsf{W}_2)$.

We claim that $\{w_i\}_{1\le i\le n}\cup\{w'_i\}_{1\le i\le n_1}\cup\{w''_i\}_{1\le i\le n_2}$ is a basis of $\mathsf{W}_1+\mathsf{W}_2$, it suffices to prove that $\{w_i\}_{1\le i\le n}\cup\{w'_i\}_{1\le i\le n_1}\cup\{w''_i\}_{1\le i\le n_2}$ is linearly independent, indeed, suppose there exist α_i, α'_j, $\alpha''_k\in F$ for $1\le i\le n$, $1\le j\le n_1$, $1\le k\le n_2$ such that

$$\sum_{i=1}^{n}\alpha_i w_i + \sum_{j=1}^{n_1}\alpha_j w'_j + \sum_{k=1}^{n_2}\alpha_k w''_k = 0,$$

equivalently,

$$v := \sum_{k=1}^{n_2}\alpha''_k w''_k = -\sum_{i=1}^{n}\alpha_i w_i - \sum_{j=1}^{n_1}\alpha'_j w'_j,$$

which implies that $v\in\mathsf{W}_1\cap\mathsf{W}_2$, then $\alpha''_1=\cdots=\alpha''_{n_2}=\alpha'_1=\cdots=\alpha'_{n_1}=\alpha_1=\cdots=\alpha_n=0$. Therefore

$$\dim(\mathsf{W}_1+\mathsf{W}_2)=\dim(\mathsf{W}_1)+\dim(\mathsf{W}_2)-\dim(\mathsf{W}_1\cap\mathsf{W}_2).$$

(b) $\Rightarrow$: From the above discussion,

$$\dim(\mathsf{W}_1+\mathsf{W}_2)=\dim(\mathsf{W}_1)+\dim(\mathsf{W}_2)-\dim(\mathsf{W}_1\cap\mathsf{W}_2),$$

since V is the direct sum of W_1 and W_2, we have

$$\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = 0,$$

therefore

$$\dim(\mathsf{W}_1 + \mathsf{W}_2) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2).$$

$\Leftarrow$: Since

$$\dim(\mathsf{W}_1 + \mathsf{W}_2) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2),$$

therefore

$$\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = 0,$$

which implies $\mathsf{W}_1 \cap \mathsf{W}_2 = 0$, then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

Exercise 14. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq n$.

(b) Prove that $\dim(\mathsf{W}_1 + \mathsf{W}_2) \leq m + n$.

Solution. (a) Let $\{w_i\}_{1 \leq i \leq l}$ be a basis of $\mathsf{W}_1 \cap \mathsf{W}_2$ where $l = \dim(\mathsf{W}_1 \cap \mathsf{W}_2)$. Since $\mathsf{W}_1 \cap \mathsf{W}_2 \subset \mathsf{W}_2$, then $\{w_i\}_{1 \leq i \leq l} \subset \mathsf{W}_2$, therefore

$$\dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq n.$$

(b) Let $\{w_i'\}_{1 \leq i \leq m}$ be a basis of W_1 and $\{w_i''\}_{1 \leq i \leq n}$ be a basis of W_2, then $\mathsf{W}_1 + \mathsf{W}_2 \subset \operatorname{span}(\{w_i'\}_{1 \leq i \leq m} \cup \{w_i''\}_{1 \leq i \leq n})$, therefore

$$\dim(\mathsf{W}_1 + \mathsf{W}_2) \leq m + n.$$

Exercise 15. (a) Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3$ with dimensions m and n, where $m > n > 0$, such that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = n$.

(b) Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3$ with dimensions m and n, where $m > n > 0$, such that $\dim(\mathsf{W}_1 + \mathsf{W}_2) = m + n$.

(c) Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3$ with dimensions m and n, where $m \geq n$, such that both $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) < n$ and $\dim(\mathsf{W}_1 + \mathsf{W}_2) < m + n$.

Solution. (a) Let $\mathsf{W}_1 := \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $\mathsf{W}_2 := \{(x, 0, 0) : x \in \mathbb{R}\}$, then $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_2) = 1$.

(b) Let $\mathsf{W}_1 := \{(x, 0, 0) : x, y, \in \mathbb{R}\}$ and $\mathsf{W}_2 := \{(0, x, 0) : x \in \mathbb{R}\}$, then $\dim(\mathsf{W}_1 + \mathsf{W}_2) = 2$.

(c) Let $\mathsf{W}_1 := \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $\mathsf{W}_2 := \{(x, 0, 0) : x \in \mathbb{R}\}$, then $\dim(\mathsf{W}_1) = 2$, $\dim(\mathsf{W}_2) = 1$ and $\dim(\mathsf{W}_1 + \mathsf{W}_2) = 2$, which implies that $\dim(\mathsf{W}_1 + \mathsf{W}_2) < \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2)$.

Exercise 16. (a) Let W_1 and W_2 be subspace of a vector space V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V.

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

Solution. (a) First, we prove that $\beta_1 \cap \beta_2 = \emptyset$. Indeed, suppose there exists $w \in \beta_1 \cap \beta_2$, then $w \in \mathsf{W}_1 \cap \mathsf{W}_2$, since $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$, therefore $w = 0$.

In addition, we prove that $\beta_1 \cup \beta_2$ is a basis for V. Since $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$, therefore for arbitrary $v \in \mathsf{V}$, there exists two unique vectors w_1 and w_2 in W_1 and W_2 respectively, which implies that $\operatorname{span}(\beta_1 \cup \beta_2) = \mathsf{V}$. Then it suffices to prove that $\beta_1 \cup \beta_2$ is a linear independent set. Indeed, let $\beta_1 = \{w_i'\}_{1 \leq i \leq |\beta_1|}$ and

$\beta_2 = \{w_i''\}_{1\le i\le|\beta_2|}$, suppose that there exist α_i', $\alpha_j'' \in F$ for $1 \le i \le |\beta_1|$, $1 \le j \le |\beta_2|$ such that

$$\sum_{i=1}^{|\beta_1|} \alpha_i' w_i' + \sum_{j=1}^{|\beta_2|} \alpha_i'' w_i'' = 0,$$

equivalently,

$$v := \sum_{i=1}^{|\beta_1|} \alpha_i' w_i' = -\sum_{j=1}^{|\beta_2|} \alpha_i'' w_i'',$$

then $v \in \mathsf{W}_1 \cap \mathsf{W}_2$, which implies that $\alpha_1' = \cdots = \alpha_{|\beta_1|}' = \alpha_1'' = \cdots = \alpha_{|\beta_2|}'' = 0$.

(b) Let $v \in \mathsf{V}$, since $\beta_1 \cup \beta_2$ is a basis for V, then there exist α_i', $\alpha_j'' \in F$ for $1 \le i \le |\beta_1|$, $1 \le j \le |\beta_2|$ such that

$$v = \sum_{i=1}^{|\beta_1|} \alpha_i' w_i' + \sum_{j=1}^{|\beta_2|} \alpha_i'' w_i'' := w_1 + w_2,$$

which implies that $\mathsf{V} = \mathsf{W}_1 + \mathsf{W}_2$.

Let $v \in \mathsf{W}_1 \cap \mathsf{W}_2$, then there exist α_i', $\alpha_j'' \in F$ for $1 \le i \le |\beta_1|$, $1 \le j \le |\beta_2|$ such that

$$v = \sum_{i=1}^{|\beta_1|} \alpha_i' w_i' = \sum_{j=1}^{|\beta_2|} \alpha_i'' w_i'',$$

which implies that

$$\sum_{i=1}^{|\beta_1|} \alpha_i' w_i' - \sum_{j=1}^{|\beta_2|} \alpha_i'' w_i'' = 0,$$

since $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V, then w_1', ..., $w_{|\beta_1|}'$, w_1'', ..., $w_{|\beta_2|}''$ are linearly independent, therefore $\alpha_1' = \cdots = \alpha_{|\beta_1|}' = \alpha_1'' = \cdots = \alpha_{|\beta_2|}'' = 0$, then $v = 0$.

Therefore $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

Exercise 17. (a) Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exits a subspace W_2 of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

(b) Let $\mathsf{V} = \mathbb{R}^2$ and $\mathsf{W}_1 = \{(a_1, 0) : a_1 \in \mathbb{R}\}$. Give examples of two different subspaces W_2 and W_2' such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$ and $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2'$.

Solution. (a) Let $\{w_i'\}_{1\le i\le n_1}$ be a basis of W_1 where $n_1 = \dim(\mathsf{W}_1)$. Since $\mathsf{W}_1 \subset \mathsf{V}$, then by Replacement theorem, there exists a linearly independent set $\{w_i''\}_{1\le i\le n_2}$ such that $\{w_i'\}_{1\le i\le n_1} \cup \{w_i''\}_{1\le i\le n_2}$ is a basis of V. Then let $\mathsf{W}_2 := \mathrm{span}(\{w_i''\}_{1\le i\le n_2} / \{w_i'\}_{1\le i\le n_1})$. Since $\{w_i'\}_{1\le i\le n_1} \cap (\{w_i''\}_{1\le i\le n_2} / \{w_i'\}_{1\le i\le n_1}) = \emptyset$, then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

(b) Let $\mathsf{W}_2 = \{(0, a_2) : a_2 \in \mathbb{R}\}$ and $\mathsf{W}_2' = \{(a, a) : a \in \mathbb{R}\}$.