MATH2040A Homework 2 suggested answer

Compulsory Part

Q1.4.10.

Solution: Denote the set of 2×2 symmetric matrices by Sym_2.

To show that $\mathrm{Span}(\{M_1, M_2, M_3\})$ is exactly Sym_2, we need to show 2 things: every such matrix can be generated by these 3 matrices ($\mathrm{Sym}_2 \subseteq \mathrm{Span}(\{M_1, M_2, M_3\})$), and every matrix generated is 2×2 symmetric ($\mathrm{Sym}_2 \supseteq \mathrm{Span}(\{M_1, M_2, M_3\})$).

We first show the $\subseteq$ direction:

Let A be a symmetric 2×2 matrix. By the assumption on A, we may assume that $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ with $a, b, c \in F$. Then $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = aM_1 + bM_2 + cM_3 \in \mathrm{Span}(\{M_1, M_2, M_3\})$. Since $A \in \mathrm{Sym}_2$ is arbitrary, $\mathrm{Sym}_2 \subseteq \mathrm{Span}(\{M_1, M_2, M_3\})$.

We now show the $\supseteq$ direction:

Let $A \in \mathrm{Span}(\{M_1, M_2, M_3\})$. By definition, we may assume that $A = aM_1 + bM_2 + cM_3 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ for some $a, b, c \in F$. Trivially, A is a 2×2 matrix, and $A^\mathsf{T} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}^\mathsf{T} = \begin{pmatrix} a & c \\ c & b \end{pmatrix} = A$, so A is also symmetric. Thus $A \in \mathrm{Sym}_2$. Since $A \in \mathrm{Span}(\{M_1, M_2, M_3\})$ is arbitrary, $\mathrm{Span}(\{M_1, M_2, M_3\}) \subseteq \mathrm{Sym}_2$.

Therefore, $\mathrm{Sym}_2 = \mathrm{Span}(\{M_1, M_2, M_3\})$.

Note

You are supposed to show that the two sets are equal, as the question is to show that "the span ... *is* the set of all 2×2 ...". If the question only asks you to show that "every matrix in the span ... is a 2×2 ..." or "every 2×2 ... is spanned by ...", then you only need to show one of the directions.

Q1.4.11.

Solution: We use the same approach of the last question. Denote $S = \{ax \mid a \in F\}$.

Let $v \in \mathrm{Span}(\{x\})$. Then by definition, $v = \sum_{i=1}^m a_i v_i$ for some positive integer m with $a_1, \ldots, a_m \in F$, $v_1, \ldots, v_m \in \{x\}$.

Since $\{x\}$ contains only one element x, $v_i = x$ for all i. So $v = \sum_{i=1}^m a_i v_i = \sum_{i=1}^m a_i x = (\sum_{i=1}^m a_i)x \in S$ as $\sum_{i=1}^m a_i \in F$. As $v \in \mathrm{Span}(\{x\})$ is arbitrary, $\mathrm{Span}(\{x\}) \subseteq S$.

On the other hand, let $v \in S$. Then $v = ax$ for some $a \in F$. As $x \in \{x\}$ and $a \in F$, $v = ax \in \mathrm{Span}(\{x\})$. As $v \in S$ is arbitrary, $S \subseteq \mathrm{Span}(\{x\})$.

Therefore, $\mathrm{Span}(\{x\}) = S = \{ax \mid a \in F\}$.

In the case of $\mathbb{R}^3$:

1. If $x = \vec{0}$, $\mathrm{Span}(\{x\}) = \{ax \mid a \in \mathbb{R}\} = \{0\}$, which is the singleton containing the origin.
2. If $x \neq \vec{0}$, $\mathrm{Span}(\{x\}) = \{ax \mid a \in \mathbb{R}\}$ is the straight line that passes through $\vec{0}$ and x.

Note

Please *do not* write something like $\text{Span}(\{x\}) = ax, a \in F$. The left-hand side is a set of vectors (the span *set*), while the right-hand side is a single vector (a multiple of x, where the scalar is arbitrarily fixed in F). Please write proper set notations.

Please note that the definition does not say anything about $\text{Span}(\{x\}) = \{ax \mid a \in F\}$. If you go back to the definition (e.g. the one in the lecture note), $\text{Span}(S)$ for nonempty S is defined as the set of all (finite) linear combinations of vectors in S (represented a sum). There is no restriction on how the vectors are chosen, and the same vector can be chosen multiple times.

You are expected to argue that each of such combinations can be represented as ax for some $a \in F$, and for all $a \in F$, ax is a linear combination of elements in $S = \{x\}$. Despite this is very straightforward, *you still need to show this.*

Q1.4.13.

Solution: Suppose $S_1 = \emptyset$. Then $\text{Span}(S_1) = \{0\}$, so $\text{Span}(S_1) = \{0\} \subseteq \text{Span}(S_2)$.

Suppose $S_1 \neq \emptyset$. Let $v \in \text{Span}(S_1)$. Then there exists a positive integer m, $a_1, \ldots, a_m \in F$, $v_1, \ldots, v_m \in S_1$ such that $v = \sum_{i=1}^m a_i v_i$.

Since $S_1 \subseteq S_2$, each of v_i is contained in S_2. So $v = \sum_{i=1}^m a_i v_i$ for some positive integer m, $a_1, \ldots, a_m \in F$, $v_1, \ldots, v_m \in S_2$. In particular, $v \in \text{Span}(S_2)$.

As $v \in \text{Span}(S_1)$ is arbitrary, $\text{Span}(S_1) \subseteq \text{Span}(S_2)$.

Suppose now that $\text{Span}(S_1) = V$. Since $S_2 \subseteq V$ and V is a vector space, $\text{Span}(S_2) \subseteq \text{Span}(V) = V$. By the above argument, we have $V = \text{Span}(S_1) \subseteq \text{Span}(S_2) = V$, so $\text{Span}(S_2) = V$.

Note

Note that $\text{Span}(S_1) = \{\sum_{i=1}^m a_i v_i \mid m \in \mathbb{Z}^+, a_1, \ldots, a_m \in F, v_1, \ldots, v_m \in S_1\}$ holds only when $S_1 \neq \emptyset$. When $S_1 = \emptyset$, there is no vector in S_1, so the set defined on the right-hand side is the empty set, which is not equal to $\text{Span}(\emptyset) = \{\vec{0}\}$. (Refer to this answer or this question on Math Stack Exchange for a quick refresh on the set-builder notation) For the same reason, you cannot pick an element $x \in S_1$ without assuming $S_1 \neq \emptyset$.

Also, note that you cannot simply assume $S_1 = \{v_1, \ldots, v_n\}$ *even if you assume S_1 is nonempty*. Such assumption implies that S_1 is a finite set (unless you specify an appropriate n), which requires an additional assumption.

Q1.4.14.

Solution: If S_1 is empty, then $\text{Span}(S_1 \cup S_2) = \text{Span}(S_2) = \{0\} + \text{Span}(S_2) = \text{Span}(S_1) + \text{Span}(S_2)$. Similarly, the equality holds if S_2 is empty. Hence in the remain part we may assume that both S_1 and S_2 are nonempty.

Let $v \in \text{Span}(S_1 \cup S_2)$. Then there exists some positive integer m such that $v = \sum_{i=1}^m a_i v_i$ for some $a_1, \ldots, a_m \in F$, $v_1, \ldots, v_m \in S_1 \cup S_2$.

For each $i \in \{1, \ldots, m\}$, $v_i \in S_1 \cup S_2$, so $v_i \in S_1$ or $v_i \in S_2$. We may without loss of generality rearrange the indices such that $v_1, \ldots, v_r \in S_1$ and $v_{r+1}, \ldots, v_m \in S_2 \setminus S_1$ for some integer $r \in \{0, \ldots, m\}$. (If none of $v_i \in S_1$ we simply take $r = 0$ and the corresponding sum in the following part $\vec{0}$; similar when none of $v_i \in S_2 \setminus S_1$)

Then $v = (\sum_{i=1}^r a_i v_i) + \left(\sum_{j=r+1}^m a_j v_j\right) \in \text{Span}(S_1) + \text{Span}(S_2)$ with $\sum_{i=1}^r a_i v_i \in \text{Span}(S_1)$, $\sum_{j=r+1}^m a_j v_j \in \text{Span}(S_2)$. As $v \in \text{Span}(S_1 \cup S_2)$ is arbitrary, $\text{Span}(S_1 \cup S_2) \subseteq \text{Span}(S_1) + \text{Span}(S_2)$.

To show the other direction, let $v \in \text{Span}(S_1) + \text{Span}(S_2)$. Then there exists $u \in \text{Span}(S_1)$, $w \in \text{Span}(S_2)$ such that $v = u + w$. By definition, there exists some positive integers m, n such that $u = \sum_{i=1}^m a_i u_i$, $w = \sum_{j=1}^n b_j w_j$ with $a_1, \ldots, a_m, b_1, \ldots, b_n \in F$, $u_1, \ldots, u_m \in S_1$, $w_1, \ldots, w_n \in S_2$.

Since $S_1 \subseteq S_1 \cup S_2$ and $S_2 \subseteq S_1 \cup S_2$, we have $v = u + w = a_1 u_1 + \ldots + a_m u_m + b_1 w_1 + \ldots + b_n w_n$ where each of $a_i, b_i \in F$, $u_i, w_i \in S_1 \cup S_2$. So $v \in \text{Span}(S_1 \cup S_2)$. As v is arbitrary, $\text{Span}(S_1) + \text{Span}(S_2) \subseteq \text{Span}(S_1 \cup S_2)$.

Therefore $\text{Span}(S_1) + \text{Span}(S_2) = \text{Span}(S_1 \cup S_2)$

Note

If you are more familiar with this notation, rather than rearranging the indices you can simply write $v = \sum_{\substack{i\in\{1,\dots,n\}\\ v_i\in S_1}} a_i v_i + \sum_{\substack{j\in\{1,\dots,n\}\\ v_j\in S_2\setminus S_1}} a_j v_j$ with the sum on empty set being $\vec{0}$ (as a convention)

Alternatively, to show that $\operatorname{Span}(S_1) + \operatorname{Span}(S_2) \subseteq \operatorname{Span}(S_1 \cup S_2)$, you can also use the result from Q1.4.13:

Solution: Since $S_1 \subseteq S_1 \cup S_2$ and $S_2 \subseteq S_1 \cup S_2$, by the result of Q1.4.13 we have $\operatorname{Span}(S_1) \subseteq \operatorname{Span}(S_1 \cup S_2)$ and $\operatorname{Span}(S_2) \subseteq \operatorname{Span}(S_1 \cup S_2)$.

Let $x \in \operatorname{Span}(S_1) + \operatorname{Span}(S_2)$. Then there exists $v_1 \in \operatorname{Span}(S_1)$, $v_2 \in \operatorname{Span}(S_2)$ such that $x = v_1 + v_2$. Since $v_1 \in \operatorname{Span}(S_1) \subseteq \operatorname{Span}(S_1 \cup S_2)$, $v_2 \in \operatorname{Span}(S_2) \subseteq \operatorname{Span}(S_1 \cup S_2)$ and $\operatorname{Span}(S_1 \cup S_2)$ is a subspace (of V), $x = v_1 + v_2 \in \operatorname{Span}(S_1 \cup S_2)$. As x is arbitrary, $\operatorname{Span}(S_1) + \operatorname{Span}(S_2) \subseteq \operatorname{Span}(S_1 \cup S_2)$.

Q1.4.15.

Solution: If $S_1 \cap S_2 = \emptyset$, $\operatorname{Span}(S_1 \cap S_2) = \{0\}$. Since $0 \in \operatorname{Span}(S_1)$ and $0 \in \operatorname{Span}(S_1)$, $0 \in \operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$ and thus $\operatorname{Span}(S_1 \cap S_2) = \{0\} \subseteq \operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$. Hence in the following proof we may assume that $S_1 \cap S_2 \neq \emptyset$. In particular both S_1, S_2 are nonempty.

Let $v \in \operatorname{Span}(S_1 \cap S_2)$. Then there exists some positive integer m such that $v = \sum_{i=1}^m a_i v_i$ for some $a_1, \dots, a_m \in F$, $v_1, \dots, v_m \in S_1 \cap S_2$. In particular, $v = \sum_{i=1}^m a_i v_i \in \operatorname{Span}(S_1)$ as for each i, $v_i \in S_1$. Similarly, $v \in \operatorname{Span}(S_2)$. Thus $v \in \operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$.

As v is arbitrary, $\operatorname{Span}(S_1 \cap S_2) \subseteq \operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$.

For the examples, we can construct two examples as follow: consider $V = \mathbb{R}^2$ is the (usual) real plane,

1. For $S_1 = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$, $S_2 = \left\{ \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$, we can show that $\operatorname{Span}(S_1 \cap S_2) = \operatorname{Span}\left(\left\{ \begin{pmatrix}1\\1\end{pmatrix} \right\}\right) = \left\{ \begin{pmatrix}x\\x\end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \subsetneq \mathbb{R}^2 = \mathbb{R}^2 \cap \mathbb{R}^2 = \operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$
2. For $S_1 = \left\{ \begin{pmatrix}1\\0\end{pmatrix} \right\}$, $S_2 = \left\{ \begin{pmatrix}0\\1\end{pmatrix} \right\}$, we can show that $\operatorname{Span}(S_1 \cap S_2) = \operatorname{Span}(\emptyset) = \{0\} = \left\{ \begin{pmatrix}x\\0\end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \cap \left\{ \begin{pmatrix}0\\x\end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} = \operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$

Note

Please also specify the base vector space in your construction of examples.

Alternatively, you can use the result from Q1.4.13:

Solution: Since $S_1 \cap S_1 \subseteq S_1$, by Q1.4.13 we have $\operatorname{Span}(S_1 \cap S_2) \subseteq \operatorname{Span}(S_1)$. Similarly, we have $\operatorname{Span}(S_1 \cap S_2) \subseteq \operatorname{Span}(S_2)$. Therefore, $\operatorname{Span}(S_1 \cap S_2) \subseteq \operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$.

The example can be constructed as above and is omitted here.

Q1.5.9.

Solution: Suppose $\{u, v\}$ are linearly dependent. Then there exists $a, b \in F$ not all zero such that $au + bv = 0$. If $a \neq 0$, we then have $u = -\frac{b}{a}v$ and so u is a multiple of v. Similarly v is a multiple of u if $b \neq 0$. Thus u or v is a multiple of the other.

Suppose on the other hand that u or v is a multiple of the other. Without loss of generality we may assume that u is a multiple of v. Then there exists $\lambda \in F$ such that $u = \lambda v$. Then $1u - \lambda v = 0$ with the coefficients not all zero. So $\{u, v\}$ is linearly dependent.

Therefore, $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Note

To show the linear dependency of a (nonempty) set S of vectors, you would only need to find one (finite) subset of vectors that have a nontrivial linear relation, i.e. find some $v_1, \ldots, v_n \in S$ and scalars $a_1, \ldots, a_n$ such that

1. some of a_i is not zero
2. $a_1v_1 + \ldots + a_nv_n = 0$

On the other hand, to show a (nonempty) set S of vectors is linearly independent, you have to show that *no such nontrivial linear relation exists.* In the case when $S = \{v_1, \ldots, v_n\}$ is finite, the proof using the very first definition usually (but not always) follows this format:

> Let $a_1, \ldots, a_n \in F$ be scalars such that $a_1v_1 + \ldots + a_nv_n = 0$.
>
> (Something related to the properties of S)
>
> So $a_1 = \ldots = a_n = 0$ is the only possible choice. By definition $\{v_1, \ldots, v_n\}$ is linearly independent.

Showing that $a_1v_1 + \ldots + a_nv_n = 0$ for $a_1 = \ldots = a_n = 0$ *does not* constitute a proof of linear independence. *This holds for all choices of $v_1, \ldots, v_n$ as it follows directly from the axioms of vector space.*

Please also be aware of the edge case where one of u, v is the zero vector. Your proof should cover this case too.

Q1.5.13.

Solution:

(a) Suppose $\{u, v\}$ is linearly independent. Let $a, b \in F$ such that $a(u+v)+b(u-v) = 0$. Then $(a+b)u+(a-b)v = 0$. Since $\{u, v\}$ is linearly independent, $a + b = a - b = 0$. Solving the system we have that $a = b = 0$ is the only solution. So $\{u + v, u - v\}$ is linearly independent.

Suppose on the other hand that $\{u + v, u - v\}$ is linearly independent. Let $a, b \in F$ such that $au + bv = 0$. Then $\frac{a+b}{2}(u + v) + \frac{a-b}{2}(u - v) = 0$. Since $\{u + v, u - v\}$ is linearly independent, $\frac{a+b}{2} = \frac{a-b}{2} = 0$. Solving the system we have that $a = b = 0$ is the only solution. So $\{u, v\}$ is linearly independent.

Therefore, $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

(b) Suppose $\{u, v, w\}$ is linearly independent. Let $a, b, c \in F$ such that $a(u+v)+b(u+w)+c(v+w) = 0$. Then $(a+b)u+(a+c)v+(b+c)w = 0$. Since $\{u, v, w\}$ is linearly independent, $a+b = a+c = b+c = 0$. Solving the system we have that $a = b = c = 0$ is the only solution. So $\{u + v, u + w, v + w\}$ is linearly independent.

Suppose on the other hand that $\{u + v, u + w, v + w\}$ is linearly independent. Let $a, b, c \in F$ such that $au + bv + cw = 0$. Then $\frac{a+b-c}{2}(u + v) + \frac{a+c-b}{2}(u + w) + \frac{b+c-a}{2}(v + w) = 0$. Since $\{u + v, u + w, v + w\}$ is linearly independent, $\frac{a+b-c}{2} = \frac{a+c-b}{2} = \frac{b+c-a}{2} = 0$. Solving the system we have that $a = b = c = 0$ is the only solution. So $\{u, v, w\}$ is linearly independent.

Therefore, $\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent.

Note

For the first part, you can also use the result from Q1.5.9 and show that no vector in the set is a multiple of the other.

You cannot show the linear independence of $\{u, v\}$ by considering only the relation of form $(a+b)u + (a-b)v = 0$ with $a, b \in F$ then reducing it to $a(u+v)+b(u-v) = 0$, unless you have also shown that the map $(a, b) \mapsto (a+b, a-b)$ is surjective from F^2 to F^2.

In another word, knowing that $a(u+v) + b(u-v) = 0$ implies $a = b = 0$ does *not* mean that you can simply rewrite $(a+b)u+(a-b)v = a(u+v)+b(u-v)$ and conclude that u, v is linearly independent because $a+b = a-b = 0$ with $a = b = 0$. You would also need to show that $\{ (a+b)u+(a-b)v \mid a, b \in F \}$ contains all possible linear combinations of u, v, i.e. $\{ (a+b)u + (a-b)v \mid a, b \in F \} = \{ cu + dv \mid c, d \in F \}$. Similar for $\{u, v, w\}$.

For example, let us consider $V = \mathbb{R}$ as a $\mathbb{R}$-vector space, and $\vec{v} = 1, \vec{u} = -1$ are vectors in V. Suppose I somehow obtained the linear relation $a^2 \cdot \vec{v} + (-b^2) \cdot \vec{u} = 0$ with $a, b \in \mathbb{R}$. Even when the only solution for this equation is $a = b = 0$, I still cannot conclude that $\{\vec{v}, \vec{u}\}$ is linearly independent as (trivially) $\{\vec{v}, \vec{u}\}$ is linearly dependent with $1 \cdot \vec{v} + 1 \cdot \vec{u} = \vec{0}$: the witnessing choice of coefficients $(1, 1)$ (or in general (λ, λ) with $\lambda \in \mathbb{R} \setminus \{0\}$) is not contained in the set $\left\{ (a^2, -b^2) \;\middle|\; a, b \in \mathbb{R} \right\}$ of coefficients considered by the linear relation.

Also, just because $a = b = 0$ makes $a(u+v) + b(u-v) = 0$ and $\{u+v, u-v\}$ is linearly independent, it does not mean that $\{u, v\}$ is also linearly independent when you plug $a = b = 0$ into $a(u+v)+b(u-v) = (a+b)u+(a-b)v$ and get a 0 with the coefficients also 0.

Q1.5.15.

Solution: Suppose S is linearly dependent. Suppose further that $u_1 \neq 0$. Then there exists $a_1, \ldots, a_n \in F$ not all zero such that $a_1u_1 + \ldots + a_nu_n = 0$.

Let $k \in \{1, \ldots, n-1\}$ be the largest index such that $a_{k+1} \neq 0$, $a_{k+2} = \ldots = a_n = 0$. k is well-defined as $u_1 \neq 0$ and n is finite. Then $a_1u_1 + \ldots + a_ku_k + a_{k+1}u_{k+1} = 0$. So $u_{k+1} = -\frac{a_1}{u_{k+1}}a_1 - \ldots - \frac{a_k}{u_{k+1}}a_k \in \operatorname{Span}(\{u_1, \ldots, u_k\})$.

Thus $u_1 = 0$ or $u_{k+1} \in \operatorname{Span}(\{u_1, \ldots, u_k\})$ for some $k \in \{1, \ldots, n-1\}$.

Suppose on the other hand that $u_1 = 0$ or $u_{k+1} \in \operatorname{Span}(\{u_1, \ldots, u_k\})$ for some $k \in \{1, \ldots, n-1\}$. If $u_1 = 0$ then S is trivially linearly dependent.

If $u_{k+1} \in \operatorname{Span}(\{u_1, \ldots, u_k\})$ for some $k \in \{1, \ldots, n-1\}$, then there exists $a_1, \ldots, a_k \in F$ not all zero such that $u_{k+1} = a_1u_1 + \ldots + a_ku_k$. In particular, $a_1u_1 + \ldots + a_ku_k - 1u_{k+1} = 0$ with the coefficients not all zero. Since $u_1, \ldots, u_{k+1} \in S$, S is linearly dependent.

Therefore S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \operatorname{Span}(\{u_1, \ldots, u_k\})$ for some $k \in \{1, \ldots, n-1\}$.

Note

For the "only if" part, the logical equivalence $(P \to (Q \vee R)) \iff ((P \wedge \neg Q) \to R)$ is used here. You can also show the proposition by enumerating cases that give the desired results, or prove by contradiction with Theorem 1.7 (only works when S is a finite set).

Alternatively, rather than considering a logically equivalent statement, we show the "only if" part directly by enumerating the possible cases:

Solution: Suppose S is linearly dependent. Then there exists scalars $a_1, \ldots, a_n \in F$ not all zero such that $a_1u_1 + \ldots + a_nu_n = 0$.

1. Consider the case where $a_n \neq 0$. Then $u_n = -\frac{a_1}{a_n}u_1 - \ldots - \frac{a_{n-1}}{a_n}u_{n-1} \in \operatorname{Span}(\{u_1, \ldots, u_{n-1}\})$.
2. Consider the case where $a_2, \ldots, a_n$ are all zero. Then $a_1 = 0$, and the relation reduces to $a_1u_1 = 0$. Since $a_1 \neq 0$, we must have $u_1 = 0$.
3. Consider the case where $a_2 \neq 0$ and $a_3, \ldots, a_n$ are all zero. Then $a_1u_1+a_2u_2 = 0$, so $u_2 = -\frac{a_1}{a_2}u_1 \in \operatorname{Span}(\{u_1\})$.
4. Consider the case where $a_3 \neq 0$ and $a_4, \ldots, a_n$ are all zero. Then $a_1u_1 + a_2u_2 + a_3u_3 = 0$ with $a_3 \neq 0$, so $u_3 = -\frac{a_1}{a_3}u_1 - \frac{a_2}{a_3}u_2 \in \operatorname{Span}(\{u_1, u_2\})$.

By enumerating all remaining cases where $a_k \neq 0$ and $a_{k+1}, \dots, a_n$ are all zero for $k \in \{4, \dots, n-1\}$ similarly, we can show that $u_k \in \operatorname{Span}(\{u_{k+1}, \dots, u_n\})$. Since one of these cases must be true, the proposition is proven.

Alternatively, you can show the "only if" part by contradiction using Theorem 1.7 in the textbook:

Solution: Suppose S is linearly dependent. We assume that $u_1 \neq 0$ and for all $k \in \{1, \dots, n-1\}$, $u_{k+1} \notin \operatorname{Span}(\{u_1, \dots, u_k\})$.

Since $u_1 \neq 0$, $\{u_1\}$ is linearly independent.

Since $u_2 \notin \operatorname{Span}(\{u_1\})$, by Theorem 1.7 $\{u_1, u_2\} = \{u_1\} \cup \{u_2\}$ is linearly independent.

Suppose $\{u_1, \dots, u_k\}$ is linearly independent and $u_{k+1} \notin \operatorname{Span}(\{u_1, \dots, u_k\})$ for some $k \in \{1, \dots, n-1\}$. Then by Theorem 1.7, we have $\{u_1, \dots, u_{k+1}\} = \{u_1, \dots, u_k\} \cup \{u_{k+1}\}$ is also linearly independent.

By induction, $\{u_1, \dots, u_k\}$ is linearly independent for all $k \in \{1, \dots, n\}$. In particular, $S = \{u_1, \dots, u_n\}$ is also linearly independent. Contradiction arises as S is linearly dependent.

So $u_1 = 0$, or for some $k \in \{1, \dots, n-1\}$, $u_{k+1} \in \operatorname{Span}(\{u_1, \dots, u_k\})$.

Optional Part

Q1.4.1.

Solution:

(a) True

(b) False

(c) True. Refer to Theorem 1.4 and 1.5

(d) False. Some information may be lost. Refer to this question on Math Stack Exchange.

(e) True

(f) False

Q1.4.4.

Solution:

(a) Yes. $(x^3 - 3x + 5) - 3(x^3 + 2x^2 - x - 1) + 2(x^3 + 3x^3 - 1) = 0$

(b) No

(c) Yes. $(-2x^3 - 11x^2 + 3x + 2) - 4(x^3 - 2x^2 + 3x - 1) + 3(2x^3 + x^3 + 3x - 2) = 0$

(d) Yes. $-(x^3 + x^2 + 2x + 13) - 2(2x^3 - 3x^2 + 4x + 1) + 5(x^3 - x^2 + 2x + 3) = 0$

(e) No

(f) No

The proofs are standard computations and are omitted here.

Q1.4.5.

Solution:

(a) Yes. $(2,-1,1) = (1,0,2) - (-1,1,1)$

(b) No

(c) No

(d) Yes. $(2,-1,1,-3) = 2(1,0,1,-1) - (0,1,1,1)$

(e) Yes. $-x^3 + 2x^2 + 3x + 3 = -(x^3 + x^2 + x + 1) + 3(x^2 + x + 1) + (x + 1)$

(f) No

(g) Yes. $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(h) No

The proofs are standard computations and are omitted here.

Q1.5.1.

Solution:

(a) False. Refer to Q1.5.10

(b) True

(c) False. It contains no vectors with nontrivial linear relation

(d) False. Consider the subset being a singleton of a (nonzero) vector

(e) True. Note that this includes the case where the subset is empty

(f) True

Q1.5.2.

Solution:

(a) Linearly dependent. $2\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(b) Linearly independent

(c) Linearly independent

(d) Linearly dependent. $4(x^3 - x) - 3(2x^2 + 4) + 2(-2x^3 + 3x^2 + 2x + 6) = 0$

(e) Linearly dependent. $-3(1,-1,2) + 2(1,-2,1) + (1,1,4) = (0,0,0)$

(f) Linearly independent

(g) Linearly dependent. $3\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(h) Linearly independent

(i) Linearly independent

(j) Linearly dependent. $4(x^4 - x^3 + 5x^2 - 8x + 6) + 3(-x^4 + x^3 - 5x^2 + 5x - 3) - 3(x^4 + 3x^2 - 3x + 5) + (2x^4 + x^3 + 4x^2 - 8x) = 0$

The proofs are standard computations and are omitted here.

Q1.5.10.

Solution: Consider $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$. The sum of the first 2 is the last vector, so they are linearly dependent. However, it is easy to verify (omitted here) that any two vectors is linearly independent.

Q1.5.16.

Solution: Instead of the original proposition, we prove the following logically equivalent proposition:

> a set S of vectors is linearly dependent if and only if there exists a finite subset of S that is linearly dependent.

Suppose S is linearly dependent. By definition, there exists a positive integer m such that $\sum_{i=1}^{m} a_i v_i = 0$ for some $a_1, \ldots, a_m \in F$ not all zero and some $v_1, \ldots, v_m \in S$. Consider the set $S' = \{v_1, \ldots, v_m\}$. Obviously S' is a finite subset of S, and by the above relation S' is linearly dependent. Hence there exist a finite linearly dependent subset.

On the other hand, suppose there exists a finite subset of S that is linearly dependent. Since this subset cannot be empty, without loss of generality we may let this subset be $\{v_1, \ldots, v_m\} \subseteq S$ for some positive integer m. Then by definition there exists $a_1, \ldots, a_m \in F$ not all zero such that $\sum_{i=1}^{m} a_i v_i = 0$. Since each $v_i \in S$, by definition S is linearly dependent.

Therefore S is linearly dependent if and only if there exists a finite linearly dependent subset. Equivalently, S is linearly independent if and only if every finite subset is linearly independent.

Note

You can also prove this proposition without transposing.

Q1.5.18.

Solution: We prove the proposition by contradiction.

Suppose S is linearly dependent. By Q1.5.16, there exists a finite subset $S' \subseteq S$ such that S' is linearly dependent. WLOG let $S' = \{p_1, \ldots, p_m\}$ for some positive integer m. Then there exists $a_1, \ldots, a_m$ not all zero such that $\sum_{i=1}^{m} a_i p_i = 0$. By removing elements with zero a_is we may further assume that all of a_i are nonzero. By assumption, the remaining set is nonempty.

Let p_k be the polynomial in S' with the highest degree. By assumption, such p_k is unique, and all other polynomials in S' have degree smaller than p_k. This implies that $0 = \sum_{i=1}^{m} a_i p_i$ has the same degree of p_k. Contradiction arises as S contains only nonzero polynomials.

Therefore S is linearly independent.

Q1.5.20.

Solution: Let $a, b \in \mathbb{R}$ such that $af + bg = 0$. This implies that $0 = af(t) + bg(t) = ae^{rt} + be^{st}$ for all $t \in \mathbb{R}$. We consider the value of the sum function at $t = 0$ and at $t = 1$: $0 = af(0)+bg(0) = a+b$ and $0 = af(1)+bg(1) = ae^r+be^s$. This implies that $a = -b$ and $a(e^r - e^s) = 0$. As $r \neq s$, $e^r \neq e^s$, and so $a = b = 0$ is the only solution. Thus f, g are linearly independent.

Note

You can also prove this by showing that the Wronskian $W(f, g) = (s - r)e^{(r+s)t}$ is nowhere zero and use the related result, or use the result of Q1.5.9 (with which the proof would still be similar to this one).