SUGGESTED SOLUTIONS TO HOMEWORK I

1. COMPULSORY PART

Exercise 1. In any vector space V, show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in \mathsf{V}$ and any $a, b \in F$.

Solution. By the properties of addition and scalar multiplication,

$$(a+b)(x+y) = a(x+y) + b(x+y) = ax + ay + bx + by.$$

Exercise 2. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Solution. V is not a vector space over $\mathbb{R}$. Indeed, for arbitrary $a_1, c_1, c_2 \in R$ and $a_2 \in \{a_2 \in R : a_2 \neq a_2^2\}$, on the one hand, we have

$$(c_1 + c_2)(a_1, a_2) = (c_1 a_1 + c_2 a_1, a_2),$$

on the other hand, we have

$$c_1(a_1, a_2) + c_2(a_1, a_2) = (c_1 a_1 + c_2 a_1, a_2^2),$$

which implies that

$$(c_1 + c_2)(a_1, a_2) \neq c_1(a_1, a_2) + c_2(a_1, a_2).$$

Exercise 3. Let V and W be vector spaces over a field F. Let

$$\mathsf{Z} = \{(v, w) : v \in V \text{ and } w \in W\}.$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1).$$

Solution. It is straightforward to verify that Z is closed under vector addition, scalar addition and scalar multiplication with the commutative property, the associative property and the distributive property.

To find a zero element in Z, let $0_{\mathsf{Z}} = (0_{\mathsf{V}}, 0_{\mathsf{W}})$ where 0_{V} and 0_{W} are two zero elements in V and W respectively, then $0_{\mathsf{Z}} \in \mathsf{Z}$, moreover, for arbitrary $v \in \mathsf{V}$ and $w \in \mathsf{W}$,

$$0_{\mathsf{Z}} + (v, w) = (0_{\mathsf{V}} + v, 0_{\mathsf{W}} + w) = (v, w),$$

which implies that 0_{Z} is a zero element in Z.

In addition, for arbitrary $v_+ \in \mathsf{V}$ and $w_+ \in \mathsf{W}$, there exists $v_- \in \mathsf{V}$ and $w_- \in \mathsf{W}$ such that $v_+ + v_- = 0_{\mathsf{V}}$ and $w_+ + w_- = 0_{\mathsf{W}}$. Therefore for $z_+ = (v_+, w_+) \in \mathsf{Z}$, let $z_- = (v_-, w_-)$, then $z_- \in \mathsf{Z}$ and

$$z_+ + z_- = (v_+ + v_-, w_+ + w_-) = 0_{\mathsf{Z}},$$

which implies that z_- is an additive inverse of z_+.

For the identity property, we have

$$1(v, w) = (1v, 1w) = (v, w),$$

for arbitrary $v \in \mathsf{V}$ and $w \in \mathsf{W}$.

Exercise 4. Let $\mathsf{P}(F)$ denote the vector space of all polynomials with coefficients form F. Is the set $W = \{f(x) \in \mathsf{P}(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $\mathsf{P}(F)$ if $n \geq 1$ where $n \in \mathbb{N}$? Justify your answer.

Solution. W is not a subspace of $\mathsf{P}(F)$. Indeed, let $f(x) = x^n + 1$ and $g(x) = -x^n$, then $f, g \in W$, however $f + g = 1 \notin W$.

Exercise 5. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$.

Solution. $\Rightarrow$: Assume $W_1 \cup W_2$ is a subspace of V, then for arbitrary $w_1 \in W_1$ and $w_2 \in W_2$, we have $w_1 - w_2 \in W_1 \cup W_2$. If $w_1 - w_2 \in W_1$, then $w_2 = w_1 - (w_1 - w_2) \in W_1$ which implies that $W_2 \subset W_1$. If $w_1 - w_2 \in W_2$, then $w_1 = w_2 + (w_1 - w_2) \in W_2$ which implies that $W_1 \subset W_2$.

$\Leftarrow$: Without loss of generality, assume $W_1 \subset W_2$, then $W_1 \cup W_2 = W_2$ which implies that $W_1 \cup W_2$ is a subspace of V.

Exercise 6. Let F_1 and F_2 be fields, $\mathcal{F}(F_1, F_2)$ denote the set of all functions from F_1 to F_2. A function $g \in \mathcal{F}(F_1, F_2)$ is called an even function if $g(-t) = g(t)$ for each $t \in F_1$ and is called an odd function if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even functions in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$.

Solution. It suffices to note that $g_0(x) \equiv 0$ is a zero element of $\mathcal{F}(F_1, F_2)$ and g_0 is not only an even function but also an odd function.

2. OPTIONAL PART

Exercise 7. Label the following statements as true or false.

(a) Every vector space contains a zero vector.
(b) A vector space may have more than one zero vector.
(c) In any vector space, $ax = bx$ implies that $a = b$ for some x.
(d) In any vector space, $ax = ay$ implies that $x = y$ for some a.
(e) A vector in F^n may be regarded as a matrix in $M_{n \times 1}(F)$.
(f) A $m \times n$ matrix has m columns and n rows.
(g) In $\mathsf{P}(F)$, only polynomials of the same degree may be added.
(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.
(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.
(j) A nonzero scalar of F may be considered to be a polynomial in $\mathsf{P}(F)$ having degree zero.
(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S.

Solution. (a) True.

(b) False. Indeed, Suppose 0 and $0'$ are two zero elements of a vector space V, then $0 = 0 + 0' = 0'$.

(c) False. Indeed, let x be the zero element, then $ax = bx = 0$ for all a and b.
(d) False. Indeed, let $a = 0$, then $ax = ay = 0$ for all x and y.
(e) True.
(f) False. Indeed, A $m \times n$ matrix has m rows and n columns.
(g) False. Indeed, the sum of polynomials with different degree is still a polynomial in $\mathsf{P}(F)$.
(h) False. Indeed, consider $f(x) = x + 1$ and $g(x) = -x$, then $f + g = 1$ is a polynomial with degree 0.
(i) True.
(j) True.
(k) True.

Exercise 8. Let $\mathsf{V} = \{(a_1, a_2, ..., a_n) : a_i \in \mathbb{C} \text{ for } i = 1, 2, ..., n\}$; So V is a vector space over $\mathbb{C}$. Is V a vector space over the field of real numbers with he operations of coordinatewise addition and multiplication?

Solution. Yes.

Exercise 9. Let $\mathsf{V} = \{(a_1, a_2, ..., a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, ..., n\}$; So V is a vector space over $\mathbb{R}$. Is V a vector space over the field of complex numbers with he operations of coordinatewise addition and multiplication?

Solution. No. Indeed, let i be the imaginary unit, then $\mathrm{i}(a_1, a_2, ..., a_n) = (ia_1, ia_2, ..., ia_n) \notin \mathsf{V}$.

Exercise 10. Let V be the set of sequences $\{a_n\}$ of real numbers. For $\{a_n\}$, $\{b_n\} \in \mathsf{V}$ and any real number t, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}, \quad \text{and} \quad t\{a_n\} = \{ta_n\}.$$

Prove that, with these operations, V is a vector space over $\mathbb{R}$.

Solution. It is straightforward to verify that V is closed under vector addition, scalar addition and scalar multiplication with the commutative property, the associative property and the distributive property.

Since for arbitrary $\{a_n\} \in \mathsf{V}$,

$$\{0\} + \{a_n\} = \{0 + a_n\} = \{a_n\},$$

which implies that $\{0\}$ is a zero element in V.

In addition, for arbitrary $\{a_n\} \in \mathsf{V}$,

$$\{a_n\} + \{-a_n\} = \{0\},$$

which implies that $\{-a_n\}$ is an additive inverse of $\{a_n\}$.

For the identity property, we have

$$1\{a_n\} = \{1a_n\} = \{a_n\},$$

for arbitrary $\{a_n\} \in \mathsf{V}$.

Exercise 11. Label the following statements as true or false.
(a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
(b) The empty set is a subspace of every vector space.
(c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $\mathsf{W} \neq \mathsf{V}$.

(d) If V is a vector space, then the intersection of any two subsets of V is a subspace of V.

(e) A $n \times n$ diagonal matrix can never have more than n nonzero entries.

(f) The trace of a square matrix is the product of its diagonal entries.

(g) Let W be the xy-plane in $\mathbb{R}^3$; that is, $\mathsf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $\mathsf{W} = \mathbb{R}^2$.

Solution. (a) False. Indeed, see Exercise 9.

(b) False. Indeed, empty set does not contain zero element.

(c) True.

(d) False. Indeed, empty set is a subset of arbitrary vector space but it is not a vector space.

(e) True.

(f) False. Indeed, A trace of a square matrix is the sum of its diagonal entries.

(g) False. Indeed, for an arbitrary element in W, it has three components, while for an arbitrary element in $\mathbb{R}^2$, it only has two component.

Exercise 12. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

(a) $\mathsf{W} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$.

(b) $\mathsf{W} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$.

(c) $\mathsf{W} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$.

(d) $\mathsf{W} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$.

(e) $\mathsf{W} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$.

(f) $\mathsf{W} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$.

Solution. (a) Yes. It is straightforward to justify W is a subspace of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$.

(b) No. Indeed, $(0, 0, 0) \notin \mathsf{W}$.

(c) Yes. It is straightforward to justify W is a subspace of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$.

(d) Yes. It is straightforward to justify W is a subspace of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$.

(e) No. Indeed, $(0, 0, 0) \notin \mathsf{W}$.

(f) No. Indeed, $(\sqrt{3}, \sqrt{5}, 0), (0, \sqrt{6}, \sqrt{3}) \in \mathsf{W}$, but $(\sqrt{3}, \sqrt{5} + \sqrt{6}, \sqrt{3}) \notin \mathsf{W}$.

Exercise 13. Prove that a subset W of a vector space V is a subspace of V if and only if $\mathsf{W} \neq \emptyset$, and, whenever $a \in F$ and $x, y \in \mathsf{W}$, then $ax \in \mathsf{W}$ and $x + y \in \mathsf{W}$.

Solution. $\Rightarrow$: Since W is a subspace of V, then W contains a zero element, therefore $\mathsf{W} \neq \emptyset$. Moreover, W is a vector space which implies that W is closed under vector addition and scalar multiplication.

$\Leftarrow$: It suffices to prove that W has a zero element. Since $\mathsf{W} \neq \emptyset$, then there exists $x_0 \in \mathsf{W}$, which also implies that $-x_0 \in \mathsf{W}$, therefore $x_0 - x_0 \in \mathsf{W}$. Let us denote $0_{\mathsf{W}} := x_0 - x_0$. We claim that 0_{W} is a zero element in W. Indeed, for arbitrary $x \in \mathsf{W}$, we have

$$x = x + x_0 - x_0.$$

Exercise 14. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $\mathsf{W}_1 + \mathsf{W}_2$.

Solution. (a) For arbitrary $w_i \in \mathsf{W}_i$, $i = 1, 2$, then we have

$$w_i = w_i + 0_{\mathsf{W}_i},$$

where $0_{\mathsf{W}_i} \in \mathsf{W}_i$ is a zero element.

(b) Let V be a vector space contains both W_1 and W_2, then for arbitrary $w \in \mathsf{W}_1 + \mathsf{W}_2$, then there exist $w_i \in \mathsf{W}_i$ for $i = 1, 2$ such that

$$w = w_1 + w_2.$$

Since $w_i \in \mathsf{V}$, therefore $w \in \mathsf{V}$ which implies that $\mathsf{W}_1 + \mathsf{W}_2 \subset \mathsf{V}$.