Solution 7

1. (a) The feasible set is $[2,4]$. The optimal value is 5. The optimal solution is 2.

(b) $L(x,\lambda) = x^2 + 1 + \lambda(x-2)(x-4)$ and
$q(\lambda) = \min_{x\in\mathbb{R}} L(x,\lambda) = L\left(\frac{3\lambda}{\lambda+1}, \lambda\right) = -\lambda - \frac{9}{\lambda+1} + 10.$

(c) Solving $q'(\lambda^*) = 0$, we have $\lambda^* = 2$ and $q(\lambda^*) = 5$. The strong duality holds.

2. (a) There is only one feasible point $x = 0$ and the primal optimal value is 0.

The dual problem is

$$\max_{\lambda\geq 0} q(\lambda) = -\frac{1}{4\lambda}.$$

Since $q(\lambda)$ increases to 0 as $\lambda \to \infty$, there is no duality gap.

(b) There is no λ such that $-\frac{1}{4\lambda} = 0$. Hence, the dual problem has no solution.

3. (a) The fensible set is $\{(1,0)\}$. The optimal solution is $(1,0)$. The optimal value is 1. (b)

$$\begin{aligned}
2x_1 + 2\lambda_1(x_1-1) + 2\lambda_2(x_1-1) &= 0 \\
2x_2 + 2\lambda_1(x_2-1) + 2\lambda_2(x_2+1) &= 0 \\
\lambda_1\left((x_1-1)^2 + (x_2-1)^2 - 1\right) &= 0 \\
\lambda_2\left((x_1-1)^2 + (x_2+1)^2 - 1\right) &= 0 \\
(x_1-1)^2 + (x_2-1)^2 - 1 &\leq 0 \\
(x_1-1)^2 + (x_2+1)^2 - 1 &\leq 0 \\
\lambda_1, \lambda_2 &\geq 0
\end{aligned}$$

When $(x_1, x_2) = (1,0)$, there is no solution to the first equation, so there are no λ_i, λ_2 such that $x^*, (\lambda_i^*, \lambda_i^*)$ satisfy KKT conditions.