Solution 6

1. (a) $\partial I(x)=\{0\}$ for $x \in X$ and $\partial I(x)=$ for $x \notin X$.
(b) $\partial f(x)=\{0\}$ for $x=0$ and $\partial f(x)=\left\{\frac{x}{|x|_2}\right\}$ for $x \neq 0$.

2. Suppose $0 \notin \partial f\left(x_0\right)$. Then $f(x)<f\left(x_0\right)$ for some $x \in X$. By convexity and differentiability, $f(x) \geq f\left(x_0\right)+\left\langle\nabla f\left(x_0\right), x-x_0\right\rangle, s_0\left\langle\nabla f\left(x_0\right), x-x_0\right) \leq f(x)-f\left(x_0\right)<0$. Conversely, suppose $0 \in \partial f\left(x_0\right)$. Let $x \in X$. Define $g(t)=f\left((1-t) x_0+t x\right)$ for $t \in[0,1]$. Since $0 \in \partial f\left(x_0\right)$, we have $f(x) \geq f\left(x_0\right)$ for all $x \in X$, so $g(t) \geq g(0)$ for all $t \in[0,1]$. This implises $g^{\prime}(0) \geq 0$, $s_0\left\langle\nabla f\left(x_0\right), x-x_0\right\rangle=g^{\prime}(0) \geq 0$.

3. (a) Since f is a continuous function on a compact domain, f attains a minimum point at some x^*.
(b) By the chain rule, $f^{\prime}(x)=\sum_{i=1}^k L^{\prime}\left(x^T u_i\right) u_i+2 x$, so $0=\sum_{i=1}^k x_i u_i+2 x$, for some $w_i \in \mathbb{R}$, so $x \in \operatorname{span}\left\{w_1, \ldots, u_k\right\}$.