Solution 5

1.

(a) Let x_0 be a minimizer of f. Then we have $f(x)-f(x_0) \geq 0 = 0^\top (x - x_0)$, so $0 \in \partial f(x_0)$.

Conversely, if $0 \in \partial f(x_0)$, then we have $f(x) - f(x_0) \geq 0^\top (x - x_0) = 0$ for all x, so x_0 is a global minimizer of f.

(b) Let $0 \in \partial f(x_0)$. Since f is convex and differentiable at x_0, we have $\partial f(x_0) = \{\nabla f(x_0)\}$, so $\nabla f(x_0) = 0$, and hence $\langle \nabla f(x_0), x - x_0 \rangle \geq 0$ for all $x \in \mathbb{R}^n$.

Conversely, suppose $\langle \nabla f(x_0), x - x_0 \rangle \geq 0$ for all $x \in \mathbb{R}^n$. Since f is convex and differentiable at x_0, we have $f(x) - f(x_0) \geq \langle \nabla f(x_0), x - x_0 \rangle \geq 0$ for all $x \in \mathbb{R}^n$, so x_0 is a global minimizer, and hence $0 \in \partial f(x_0)$.

2. Please refer to Proposition 2.6 in Note 1.

3. Please refer to Remark 2.1 (following Theorem 2.4) in Note 1.